

A brief note on tensor product of Hilbert spaces

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Abstract

Here we consider the tensor products of infinite-dimensional Hilbert spaces.

1 Preliminaries

In this note we are concerned with tensor products of Hilbert spaces. We focus our attention to real Hilbert spaces, though the developments extend easily to the complex case. Here we adopt the direct approach used in [1]. Let \mathcal{H}_1 and \mathcal{H}_2 be real Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_{\mathcal{H}_1}$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}_2}$. In what follows, for convenience, we drop the subscripts indicating the dependence of the inner product to the corresponding space, and the choice of the inner product should be inferred from the context. For $\phi \in \mathcal{H}_1$ and $\eta \in \mathcal{H}_2$, we define the bilinear form $\phi \otimes \eta : \mathcal{H}_1 \times \mathcal{H}_2 \rightarrow \mathbb{R}$ by,

$$[\phi \otimes \eta](x, y) = \langle x, \phi \rangle \langle y, \eta \rangle, \quad (x, y) \in \mathcal{H}_1 \times \mathcal{H}_2.$$

Let \mathcal{E} be the set of all finite linear combinations of such bilinear forms. An inner product, $\langle\langle \cdot, \cdot \rangle\rangle$ on \mathcal{E} can be defined as follows:

$$\langle\langle \phi_1 \otimes \eta_1, \phi_2 \otimes \eta_2 \rangle\rangle = \langle \phi_1, \phi_2 \rangle \langle \eta_1, \eta_2 \rangle, \quad (1.1)$$

where $\phi_1, \phi_2 \in \mathcal{H}_1$ and $\eta_1, \eta_2 \in \mathcal{H}_2$. That $\langle\langle \cdot, \cdot \rangle\rangle$ is well defined and is positive definite can be verified directly (cf. [1] for a basic proof).

Definition 1.1. *The tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of \mathcal{H}_1 and \mathcal{H}_2 is defined as the completion of \mathcal{E} under the inner-product $\langle\langle \cdot, \cdot \rangle\rangle$ defined in (1.1).*

2 Tensor product bases

Let us consider Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 with orthonormal bases $\{\phi_i\}$ and $\{\eta_j\}$ respectively. The following basic result shows how to build a basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$. The basic proof given here follows that of [1] closely.

Theorem 2.1. *Let \mathcal{H}_1 and \mathcal{H}_2 be Hilbert spaces with orthonormal bases $\{\phi_i\}$ and $\{\eta_j\}$ respectively. Then, $\{\phi_i \otimes \eta_j\}$ is an orthonormal basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$.*

Proof. That $\{\phi_i \otimes \eta_j\}$ is an orthonormal set is clear. We need to show that the closed subspace M obtained as span of $\{\phi_i \otimes \eta_j\}$ contains \mathcal{E} . Take $\phi \otimes \eta \in \mathcal{E}$, and note that,

$$\phi = \sum_i c_i \phi_i, \quad \eta = \sum_j d_j \eta_j,$$

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with $\sum_i |c_i|^2 < \infty$ and $\sum_j |d_j|^2 < \infty$. Thus, $\sum_{i,j} |c_i d_j| < \infty$, and therefore, $\sum_{i,j} |c_i d_j|^2 < \infty$. Consequently, we know that $\sum_{i < m, j < n} c_i d_j \phi_i \otimes \eta_j$ converges in M as $n, m \rightarrow \infty$, and it is straightforward to note that

$$\left\| \phi \otimes \eta - \sum_{i < m, j < n} c_i d_j \phi_i \otimes \eta_j \right\| \rightarrow 0, \quad \text{as } n, m \rightarrow \infty.$$

That is $\phi \otimes \eta \in M$. □

Remark 2.2. Consider a Hilbert space \mathcal{H} with an orthonormal basis $\{\phi_i\}$. In the proof of the above theorem, we used the fact that if $\sum_i |c_i|^2 < \infty$ where $c_i \in \mathbb{R}$, then $\sum_i c_i \phi_i$ is in \mathcal{H} . This is straightforward to show; to do so, we proceed as follows. Note that for the sequence $\{y_n\}$ defined by, $y_n = \sum_{i=1}^n c_i \phi_i$, we have (for $m > n$),

$$\|y_m - y_n\|^2 = \left\| \sum_{i=n+1}^m c_i \phi_i \right\|^2 = \sum_{i=n+1}^m |c_i|^2.$$

Therefore, $\{y_n\}$ is a Cauchy sequence, and since \mathcal{H} is complete, it follows that $\{y_n\}$ converges in \mathcal{H} .

3 An important example

Let (X, \mathcal{A}, μ) be a measure space. Recall that $L^2(X, \mathcal{A}, \mu)$ is the space of real valued measurable functions that are square integrable. In what follows, we consider the case that $X \subseteq \mathbb{R}^n$ and \mathcal{A} is the Borel σ -algebra on X . Therefore, we drop the σ -algebra and use the notation $L^2(X, \mu)$ for brevity. Now, suppose $\{\phi_i(x)\}$ and $\{\eta_j(y)\}$ are orthonormal bases for $L^2(\Omega_1, \mu_1)$ and $L^2(\Omega_2, \mu_2)$ respectively. Consider then the set $\{\phi_i(x)\eta_j(y)\}$. We can show that $\{\phi_i(x)\eta_j(y)\}$ is a complete orthonormal set in $L^2(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)$ as follows. First note that orthonormality is immediate. As for completeness, let $f(x, y) \in L^2(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)$ be such that

$$\int_{\Omega_2} \int_{\Omega_1} f(x, y) \phi_i(x) \eta_j(y) d\mu_1(x) d\mu_2(y) = 0, \quad \text{for all } i, j.$$

We claim that $f = 0$ (almost everywhere). We have, for all i, j ,

$$0 = \int_{\Omega_2} \int_{\Omega_1} f(x, y) \phi_i(x) \eta_j(y) d\mu_1(x) d\mu_2(y) = \int_{\Omega_2} \left(\int_{\Omega_1} f(x, y) \phi_i(x) d\mu_1(x) \right) \eta_j(y) d\mu_2(y).$$

Now using the fact that $\{\eta_j\}$ is a orthonormal basis for $L^2(\Omega_2, \mu_2)$ we have that for all i ,

$$\int_{\Omega_1} f(x, y) \phi_i(x) d\mu_1(x) = 0, \quad \mu_2\text{-almost everywhere.}$$

Let $E_i \subseteq \Omega_2$ be the set of measure zero where the above equality does not hold and let $E = \cup_i E_i$. Then, for $y \notin E$, $\int_{\Omega_1} f(x, y) \phi_i(x) d\mu_1(x) = 0$ for all i , and thus, again using the fact that $\{\phi_i\}$ is a complete orthonormal set in $L^2(\Omega_1, \mu_1)$, we have that $f(x, y) = 0$ μ_1 -almost everywhere. Therefore, $f = 0$, $(\mu_1 \otimes \mu_2)$ -almost everywhere. Hence, we have shown that $\{\phi_i(x)\eta_j(y)\}$ is a complete orthonormal set in $L^2(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)$.

We can also show the isomorphism,

$$L^2(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2) \cong L^2(\Omega_1, \mu_1) \otimes L^2(\Omega_2, \mu_2),$$

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as follows. Let us define a mapping U that takes an orthonormal basis of $L^2(\Omega_1, \mu_1) \otimes L^2(\Omega_2, \mu_2)$ onto an orthonormal basis of $L^2(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)$:

$$U(\phi_i \otimes \eta_j) = \phi_i(x)\eta_j(y).$$

And note that U extends uniquely to a unitary mapping, of $L^2(\Omega_1, \mu_1) \otimes L^2(\Omega_2, \mu_2)$ onto $L^2(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)$. Moreover, note that for $f \in L^2(\Omega_1, \mu_1)$ and $g \in L^2(\Omega_2, \mu_2)$,

$$f = \sum_i c_i \phi_i, \quad g = \sum_j d_j \eta_j,$$

and we have,

$$U(f \otimes g) = U\left(\left(\sum_i c_i \phi_i\right) \otimes \left(\sum_j d_j \eta_j\right)\right) = U\left(\sum_{i,j} c_i d_j (\phi_i \otimes \eta_j)\right) = \sum_{i,j} c_i d_j \phi_i \eta_j = \left(\sum_i c_i \phi_i\right) \left(\sum_j d_j \eta_j\right) = fg.$$

References

- [1] Michael Reed and Barry Simon. *Methods of modern mathematical physics 1: Functional Analysis*. Academic Press, 1972.