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SOME NOTES ON RIEMANN LEBESGUE LEMMA

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1. Preliminary Comments. This brief note aims to establish what is know as Riemann-Lebesgue Lemma. In fact, what we will prove, will have as a consequence what most people know as the Riemann-Lebesgue Lemma – the result that says the Fourier coefficients of an L^2 function expanded in the standard orthonormal basis $\{e^{inx}\}_{-\infty}^{\infty}$ go to zero as $n \to \infty$. The exposition is mainly based on the discussion in [1]. To keep the discussion simple, we work in an L^2 setting; the discussion that follows can easily be extended to L^p with $1 < p < \infty$. For the cases of $p = 1$ and $p = \infty$ see [1].

2. The Main Theorem. Before proving the main result, we need the following lemma.

LEMMA 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then, $f_n \rightharpoonup f$ in $L^2(\Omega)$ if and only if*

1. $\|f_n\|_2 \le M$ *for all* n.
2. $\lim_{n\to\infty} \int_D \Big[f_n(x) - f(x)\Big]\, dx = 0$ *for every cube* $D \subset \Omega$.

Proof. Without loss of generality, suppose $f = 0$. suppose $f_n \rightharpoonup 0$; then, (1) follows from the uniform boundedness principle; also, we have (2) because for any cube $D \subset \Omega$, $\chi_D \in L^2(\Omega)$ and using the weak convergence of $\{f_n\}$, $\int_D f_n(x)\, dx = \int_\Omega \chi_D(x) f_n(x)\, dx \to 0$.

Conversely, suppose (1) and (2) hold. We need to show, $\int_\Omega \phi(x) f_n(x) \to 0$ for all $\phi \in L^2(\Omega)$. Since simple functions are dense in $L^2(\Omega)$ and we can write a simple function as a finite sum of characteristic functions, we will prove the result for characteristic functions only.

Let $\phi = \chi_E$ for some measurable subset $E \subset \Omega$. Using regularity properties of Lebesgue measure, we can approximate E from inside using cubes; thus, $\forall \epsilon > 0$, there exist, $\{D_i\}_1^N$, such that $D_i \subset E$ and $\mu(E \setminus \cup_i^N D_i) < (\frac{\epsilon}{M})^2$. For convenience, denote $D := \cup_i^N D_i$ and $D' := E \setminus \cup_i^N D_i$. Then, we have

$$\begin{aligned}
\int_\Omega \chi_E f_n\, dx &= \int_E f_n\, dx \\
&= \int_D f_n\, dx + \int_{D'} f_n\, dx \\
&\le \int_D f_n\, dx + \left(\int_{D'} 1^2\, dx\right)^{\frac{1}{2}} \left(\int_{D'} f_n^2\, dx\right)^{\frac{1}{2}} \\
&\le \int_D f_n\, dx + M\mu(D')^{\frac{1}{2}} \\
&< \int_D f_n\, dx + \epsilon.
\end{aligned}$$

From which the result follows; note that we have used Cauchy-Schwarz along with boundedness of $\{f_n\}$.
□

Now, we are ready to state and prove the main result. The proof that follows is based on the more general argument in [1].

THEOREM 2.2. *Let* $\Omega = \prod_1^n (a_i, b_i)$ *and let* $f \in L^2(\Omega)$. *Extend* f *by periodicity from* Ω *to* $\mathbb{R}^n$. *Let* $f_m(x) = f(mx)$. *Then, as* $m \to \infty$, $f_m \rightharpoonup \bar{f}$ *in* $L^2(\Omega)$, *where* $\bar{f} := \frac{1}{\mu(\Omega)} \int_\Omega f(x)\,dx$.

Proof. By the previous lemma, it is sufficient to show the following to obtain the result.

1. $\|f_m\|_2 \le M$ for all m (for some $M \in \mathbb{R}$).
2. $\lim_{m\to\infty} \int_D \left[f_m(x) - \bar{f}(x)\right] dx = 0$ for every cube $D \subset \Omega$.

To show the first item, we proceed (having periodicity of f in mind) as follows:

$$\begin{aligned}\int_\Omega |f_m(x)|^2\,dx &= \int_\Omega |f(mx)|^2\,dx \\ &= \frac{1}{m^n}\int_{m\Omega} |f(x)|^2\,dx \\ &= \int_\Omega |f(x)|^2\,dx \\ &= \|f\|_2^2,\end{aligned}$$

from which (1) follows. To show (2), we first need to have a suitable representation for a typical cube D in Ω; for any such D, we can find α and β in $\mathbb{R}$ such that $D = \alpha + \beta\Omega$, where

$$\alpha + \beta\Omega = \prod_1^n (\alpha_i + \beta_i a_i, \alpha_i + \beta_i b_i).$$

We will also develop the notation $[x]$ for the integer part of x which will become useful in a minute. We proceed as follows. Let D be any arbitrary cube in Ω. Then, we have

$$\begin{aligned}\int_D (f_m(x) - \bar{f})\,dx &= \int_{\alpha+\beta\Omega} (f(mx) - \bar{f})\,dx \\ &= \frac{1}{m^n}\int_{m\alpha+[m\beta]\Omega} (f(x) - \bar{f})\,dx + \frac{1}{m^n}\int_{m\alpha+(m\beta-[m\beta])\Omega} (f(x) - \bar{f})\,dx \\ &= \frac{[m\beta]^n}{m^n}\int_\Omega (f(x) - \bar{f})\,dx + \frac{1}{m^n}\int_{m\alpha+(m\beta-[m\beta])\Omega} (f(x) - \bar{f})\,dx \\ &= \frac{1}{m^n}\int_{m\alpha+(m\beta-[m\beta])\Omega} (f(x) - \bar{f})\,dx.\end{aligned}$$

In the above calculation we used the fact that f is periodic and that $\int_\Omega (f(x) - \bar{f})\,dx = 0$. Next, we conclude,

$$|\int_D (f_m(x) - \bar{f})\,dx| = \frac{1}{m^n}|\int_{m\alpha+(m\beta-[m\beta])\Omega} (f(x) - \bar{f})\,dx| \le \frac{1}{m^n}\int_\Omega |f(x) - \bar{f}|\,dx \to 0,$$

as $m \to \infty$ and thus, (2) is proved also. Therefore, by Lemma 2.1, we have that $f_m \rightharpoonup \bar{f}$ as $m \to \infty$. ∎

3. References.

1. Dacorogna, B. Direct Methods in Calculus of Variations. Springer (1989).