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RELLICH'S THEOREM AND SOME OF ITS APPLICATIONS

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1. Preliminary Comments.

In this note, we will go through the proof of Rellich's Theorem. In general Rellich's Theorem says that the inclusion $H^{m+1}(\Omega) \hookrightarrow H^m(\Omega)$ is compact (Ω is a bounded domain). We will show the result for the case of $H_0^1(\Omega)$; that is we will establish, $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Next, we will use properties of compact operators to get some corollaries of the Theorem. The proof of Rellich's Theorem requires several technical developments which will be discussed in the next Section.

2. Technical Tools Needed in the Proof of Rellich's Theorem.

In this section we will present the technical tools needed in the proof of Rellich's Theorem. We start by some definitions.

Definition 2.1. *(Relatively Compact)*
Let X be a metric space; $A \subseteq X$ is relatively compact in X, if $\bar{A}$ is compact in X.

Definition 2.2. *(Precompact)*
Let X be a metric space; $A \subseteq X$ is precompact (also called totally bounded) if for every $\epsilon > 0$, there exist finitely many points $x_1, \ldots, x_N$ in A such that $\cup_1^N B(x_i, \epsilon)$ covers A.

The following Theorem shows that when we are working in a complete metric space, precompactness and relative compactness are equivalent.

Theorem 2.3. *Let X be a metric space. If $A \subseteq X$ is relatively compact then it is precompact. Moreover, if X is complete then the converse holds also.*

The following results which are presented without proof are the main tools in proving Rellich's Theorem; their proofs can be found in [1]. However, apart from the first Theorem, the proof of the following Lemmas is not too involved and amounts to some technical details.

The following Theorem provides a criterion for relative compactness of a set in L^p [1].

Theorem 2.4. *Let $A \subset L^p$. The A is relatively compact in L^p if and only if*

1. *A is bounded in L^p.*
2. $\displaystyle\lim_{R\to\infty} \int_{\{|x|>R\}} |f(x)|^p \, dx = 0$ *uniformly with respect to $f \in A$.*
3. $\displaystyle\lim_{a\to 0} \tau_a f = f$ *uniformly with respect to $f \in A$.*

Note that in the above Theorem, $\tau_a f(x) := f(x-a)$. The proof of above Theorem is rather technical; a clear and concise proof is provided in [1].

The following two Lemmas are also needed.

Lemma 2.5. *Let Ω be a bounded domain and let $f \in H_0^1(\Omega)$. Define,*

$$\tilde{f} := \begin{cases} f & on \quad \Omega \\ 0 & on \quad \mathbb{R}^d \setminus \Omega \end{cases}$$

Then $\tilde{f} \in H^1(\mathbb{R}^d)$ and $\Phi : \left(H_0^1(\Omega), \|.\|_{H_0^1(\Omega)}\right) \to \left(H^1(\mathbb{R}^d), \|.\|_{H^1(\mathbb{R}^d)}\right)$ is an isometry.

Lemma 2.6. *Let $f \in H^1(\mathbb{R}^d)$ the for every $h \in \mathbb{R}^d$*

$$\|\tau_h f - f\|_{L^2} \leq |h| \|\nabla f\|_{L^2}.$$

3. Rellich's Theorem.

Since we are discussing compact imbeddings, first we take the time to define a compact operator formally.

DEFINITION 3.1. *Let X and Y be two normed linear spaces and $T : X \to Y$ a linear map between X and Y. T is called a compact operator if for all bounded sets $E \subseteq X$, $T(E)$ is relatively compact in Y.*

We are now ready to state and prove Rellich's Theorem. Note that the following proof is structured in the same spirit as the discussion of Rellich's Theorem in [1].

THEOREM 3.2. *(Rellich) Let Ω be a bounded domain in $\mathbb{R}^d$; then the inclusion map $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is a compact operator.*

Proof. First define the map, $\Phi : L^2(\mathbb{R}^d) \to L^2(\Omega)$ given by $\Phi(u) = u \mid_\Omega$ and note that Φ is clearly continuous; next, we define $\Xi : H_0^1(\Omega) \to L^2(\mathbb{R}^d)$ by $\Xi(f) = \tilde{f}$, where $\tilde{f}$ is the extension of f as defined in Lemma 2.5. We want to show, $\mathcal{I} : H_0^1(\Omega) \to L^2(\Omega)$ is compact, where $\mathcal{I} f = f$; we can use $\mathcal{I} = \Phi \circ \Xi$. Also, we note that the image of relatively compact set under a continuous mapping between Banach spaces is again relatively compact; hence, it is enough to prove that $\Xi : f \to \tilde{f}$ from $H_0^1(\Omega)$ to $L^2(\mathbb{R}^d)$ is a compact operator.

We know by Lemma 2.5 that Ξ is an isometry such that $\|f\|_{H_0^1(\Omega)} = \|\Xi(f)\|_{H^1(\mathbb{R}^d)}$. Let B be the closed unit ball in $H_0^1(\Omega)$. Define, $\tilde{B} := \Xi(B) = \{\tilde{f} \mid f \in B\}$, and note that by Lemma 2.5 $\tilde{B}$ is contained in the closed unit ball of $H^1(\mathbb{R}^d)$. If we show that $\tilde{B}$ is relatively compact in $L^2(\mathbb{R}^d)$ we are done; to do so we appeal to the criterion for relative compactness in L^p provided by Theorem 2.4. In what follows we will show that items (1), (2), and (3) of Theorem 2.4 are satisfied.

Boundedness of $\tilde{B}$ in $L^2(\mathbb{R}^d)$ was established above where we noted that $\tilde{B}$ is in the closed units ball of $H^1(\Omega) \subset L^2(\mathbb{R}^d)$; from this we get that for any $f \in B$, $\|\tilde{f}\|_{L^2} \leq \|\tilde{f}\|_{H^1} \leq 1$. Moreover, for any $R > 0$ such that $\Omega \subset B(0, R)$, we have

$$\int_{\{|x|>R\}} |\tilde{f}(x)|^2 \, dx = 0,$$

and hence follows the item (2) of Theorem 2.4. Thus, it remains to show item (3) of the aforementioned Theorem. Recall that by lemma 2.6, we have for $\tilde{f} \in \tilde{B}$

$$\|\tau_h \tilde{f} - \tilde{f}\|_{L^2} \leq |h| \| \, |\nabla \tilde{f}| \, \|_{L^2(\mathbb{R}^d)} \leq |h| \|\tilde{f}\|_{H^1(\mathbb{R}^d)} \leq |h|,$$

from which we have $\|\tau_h \tilde{f} - \tilde{f}\|_{L^2} \to 0$ as $h \to 0$. Hence, we also have property (3) of Theorem 2.4. Therefore, $\tilde{B}$ is relatively compact in $L^2(\mathbb{R}^d)$; this completes the proof. □

4. Some Implications of Rellich's Lemma.

The following is a basic Theorem regarding compact operators.

THEOREM 4.1. *Let X and Y be two normed linear spaces; suppose $T : X \to Y$, is a linear operator. Then the following are equivalent.*

1. *T is compact.*
2. *The image of the open unit ball under T is relatively compact in Y.*
3. *For any bounded sequence $\{x_n\}$ in X, there exist a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ that converges in Y.*

Using the properties of compact operators and Rellich's Theorem we conclude that any bounded set in $H_1^0(\Omega)$ is relatively compact in $L^2(\Omega)$. In particular, the following corollary is very useful in applications.

Corollary 4.2. *Let Ω be a bounded domain in $\mathbb{R}^d$. Any bounded sequence in $H_0^1(\Omega)$ has a subsequence that converges strongly in $L^2(\Omega)$.*

To conclude this brief note, we recall that what we proved is a special case of Rellich's Theorem; for a discussion of the Theorem in greatest generality, Adams [2] is a good reference. A last remark would be to comment that Rellich's Theorem is indeed a very deep result based on some fundamental Theorems in analysis and measure Theory; the proof of Theorem 2.4 uses one major result for each direction of the proof: one implication (that A is precompact implies (1), (2), and (3)) uses Lebesgue Dominated Convergence Theorem, whereas the converse implication uses Ascolli's Theorem [1].

References.

1. Hirsch, F. and Lacombe, G. Elements of Functional Analysis. Springer (1999)

2. Adams, R. A. Sobolev Spaces. Academic Press (1975)