A basic note on convergence of quadrature formulas

Alen Alexanderian*

Abstract

We discuss a fundamental result on convergence of quadrature formulas.

We consider a family of quadrature formulas

$$I_n(f) = \sum_{j=1}^{n} w_j^n f(x_j^n),$$

that approximate $I(f) = \int_a^b f(x)\,dx$, where $x_j^n \in [a,b]$ and $w_j^n \in \mathbb{R}$, $j \in \{1,\ldots,n\}$ are the nodes and weights of the quadrature formula, and we have used superscript n to indicate that the nodes and weights depend on a given n (here the superscripts do not denote exponents). In this note, we assume $[a,b]$ is a given finite interval.

1 A basic result

The following result [1] provides necessary and sufficient conditions for convergence of $I_n(f)$ to $I(f)$, as $n \to \infty$.

Theorem 1.1. *Let $I_n(f) = \sum_{j=1}^n w_j^n f(x_j^n)$, $n \geq 1$, be a sequence of quadrature formulas that approximate $I(f)$. Let $\mathscr{F}$ be a family of continuous functions on $[a,b]$ that is dense in $C[a,b]$.*[1] *Then,*

$$\lim_{n\to\infty} I_n(f) = I(f), \quad \text{for all } f \in C[a,b], \tag{1.1}$$

if and only if the following hold:

$$I_n(f) \to I(f), \quad \text{for all } f \in \mathscr{F}, \tag{1.2a}$$

and

$$B := \sup_{n\geq 1} \sum_{j=1}^{n} |w_j^n| < \infty. \tag{1.2b}$$

Proof. Assuming (1.1) holds, we immediately get (1.2a). Showing (1.1) implies (1.2b) requires an application of Banach–Steinhaus Theorem. A proof of this implication is provided later below.

Next, we prove that (1.2a) and (1.2b) imply (1.1). Let $f \in C[a,b]$, and let $\varepsilon > 0$ be fixed but arbitrary. Since $\mathscr{F}$ is dense in $C[a,b]$ there exists $f_\varepsilon \in \mathscr{F}$ such that.

$$\max_{x\in[a,b]} |f(x) - f_\varepsilon(x)| \leq \frac{\varepsilon}{2(b-a+B)}.$$

We note

$$\begin{aligned} |I(f) - I_n(f)| &= |I(f) - I(f_\varepsilon) + I(f_\varepsilon) - I_n(f_\varepsilon) + I_n(f_\varepsilon) - I_n(f)| \\ &\leq |I(f) - I(f_\varepsilon)| + |I(f_\varepsilon) - I_n(f_\varepsilon)| + |I_n(f_\varepsilon) - I_n(f)|. \end{aligned} \tag{1.3}$$

Considering the first and third terms in right hand side of (1.3) we have,

$$|I(f) - I(f_\varepsilon)| \leq \int_a^b |f(x) - f_\varepsilon(x)|\,dx \leq \frac{\varepsilon(b-a)}{2(b-a+B)}, \tag{1.4}$$

*North Carolina State University, Raleigh, NC, USA. E-mail: alexanderian@ncsu.edu

Last revised: March 5, 2018..

[1] Recall that a subset S of a normed linear space $(X, \|\cdot\|)$ is dense in X if for every $v \in X$ and every $\varepsilon > 0$, there exists a $v_\varepsilon \in S$ for which $\|v - v_\varepsilon\| \leq \varepsilon$.

and

$$|I_n(f_\varepsilon) - I_n(f)| = |\sum_{j=1}^{n} w_j^n f_\varepsilon(x_j^n) - \sum_{j=1}^{n} w_j^n f(x_j^n)| \leq \sum_{j=1}^{n} |w_j^n||f_\varepsilon(x_j^n) - f(x_j^n)| \leq \frac{B\varepsilon}{2(b-a+B)}. \quad (1.5)$$

Moreover, since $f_\varepsilon \in \mathscr{F}$, we have by (1.2a) that there exists N_ε such that

$$|I(f_\varepsilon) - I_n(f_\varepsilon)| \leq \frac{\varepsilon}{2}, \quad \text{for all } n \geq N_\varepsilon. \quad (1.6)$$

Therefore, using (1.4), (1.5), and (1.6), in (1.3), we have, for every $n \geq N_\varepsilon$,

$$|I(f) - I_n(f)| \leq \frac{\varepsilon(b-a)}{2(b-a+B)} + \frac{\varepsilon}{2} + \frac{B\varepsilon}{2(b-a+B)} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have that $I_n(f) \to I(f)$ as $n \to \infty$.

□

2 Proof of the implication (1.1) ⇒ (1.2b)

First we recall the following well known result from functional analysis, which is known as the Uniform Boundedness Principle or Banach–Steinhaus Theorem; see e.g., [2].

Theorem 2.1 (Uniform Boundedness Principle)**.** *Let X be a Banach space and Y be a normed linear space. Let $\{T_n\}_{n=1}^\infty$ be a sequence of bounded linear transformations $T_n : X \to Y$, such that $\{T_n x\}_{n=1}^\infty$ is bounded for every x. That is, for every $x \in X$, there exists a real number M_x such that,*

$$\|T_n x\| \leq M_x, \quad \text{for all } n \geq 1.$$

Then, the sequence of operator norms $\{\|T_n\|\}_{n=1}^\infty$ is bounded. That is, there exists a real number M such that

$$\|T_n\| \leq M, \quad \text{for all } n \geq 1.$$

We next fill the gap in proof of Theorem 1.1.

Proof of (1.1) ⇒ (1.2b)*:* Note that for each n, $I_n : C[a,b] \to \mathbb{R}$, where $C[a,b]$ is equipped with the uniform norm $\|\cdot\|_\infty$, is a bounded linear functional. This is seen easily through

$$|I_n(f)| = |\sum_{j=1}^{n} w_j^n f(x_j^n)| \leq \big(\sum_{j=1}^{n} |w_j^n|\big)\|f\|_\infty.$$

It is straightforward to show (do it as exercise) that the operator norm of $I_n(f)$ is given by $\|I_n\| = \sum_{j=1}^n |w_j^n|$. Now, assuming (1.1) we immediately get that $|I_n(f)|$ is bounded for every $f \in C[a,b]$. Therefore, applying the Uniform Boundedness Principle, we get that $\{\|I_n\|\}_{n=1}^\infty$ is bounded. That is there exists a real number M such that

$$\sum_{j=1}^{n} |w_j^n| = \|I_n\| \leq M, \quad \text{for all } n \geq 1,$$

and hence (1.2b) follows. □

References

[1] Kendall E. Atkinson. *An introduction to numerical analysis*. John Wiley & Sons Inc., New York, second edition, 1989.

[2] Kreyszig, Erwin. Introductory functional analysis with applications. 1978.