On Stability of Linear Complementarity Systems

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Abstract

A Linear Complementarity System (LCS) is a special type of a dynamical system which involves a system of Ordinary Differential Equations (ODEs) coupled with a Linear Complementarity Problem (LCP). In this note, we provide a careful study of some new developments in stability theory of Linear Complementarity Systems. After discussing some background results from LCP theory and ODE theory, we will discuss an extension of the well known LaSalle's Theorem in the context of an LCS developed by the authors in [2].

Contents

1 Introduction

LCSs comprise a special class of dynamical systems which are defined by a linear ODE system of form

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \\ \mathbf{x}(0) = \mathbf{x}^0, \end{cases}$$

where A and B are constant matrices and $\mathbf{u}$ is an algebraic variable which is the solution to the LCP:

$$0 \le \mathbf{u} \perp C\mathbf{x} + D\mathbf{u} \ge 0,$$

where C and D are constant matrices. Hence, we see that an LCS involves a linear ODE system coupled with an LCP.

In general, given constant matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{m\times m}$, LCS$(A, B, C, D)$ is the problem of finding the state trajectory $\mathbf{x} = \mathbf{x}(t) \in \mathbb{R}^n$ and control input $\mathbf{u} = \mathbf{u}(t) \in \mathbb{R}^m$ such that

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \\ 0 \leq \mathbf{u} \perp C\mathbf{x} + D\mathbf{u} \geq 0, \\ \mathbf{x}(0) = \mathbf{x}^0. \end{cases} \tag{1}$$

Naturally, LCS theory borrows many elements from LCP theory and ODE systems theory. A detailed discussion of some fundamental issues in LCS theory can be found in [8]. There are also survey papers such as [6] which provide examples of the types of problems addressed in context of Complementarity Systems and Linear Complementarity Systems.

In this technical note, we will be discussing some recent developments in stability theory of LCSs given in [2]. Our focus will be mainly on an extension of LaSalle's Theorem (from stability theory for non-linear ODEs) presented in the aforementioned paper. We will discuss, in detail, the case where the matrix D in (1) is a P-matrix; extension to non-P case which is also covered in [2] will be mentioned briefly.

The organization of this paper is as follows. In the Section 2, we will discuss some background results from LCP theory, theory of multi-valued functions, and some stability results including LaSalle's Theorem. In Section 3, we will cover the stability results for LCSs. Finally, we will close our discussion by some concluding remarks in Section 4

2 Some Theoretical Background

In this Section, we will review some basic ideas from LCP theory, theory of set-valued maps, and theory of ODE systems. The results stated in this section, and the notation developed herein will be used extensively in Section 3 where we will be discussing stability results for the case of an LCS. Note that the results presented in this Section are only the ones needed in our subsequent discussion on LCSs.

2.1 Concepts from LCP theory

Here, we recall some basic results from the theory of Linear Complementarity Problems and set valued mappings. The results presented in this section, most of which presented without proof, will be essential in our subsequent developments. We start by first developing some notation.

For vectors $\mathbf{u}$ and $\mathbf{v}$ belonging to the Euclidean space $\mathbb{R}^n$, we say $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\sum_{i=1}^n u_i v_i = 0$; we use the notation $\mathbf{u} \perp \mathbf{v}$ to say $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. Given a vector $\mathbf{q} \in \mathbb{R}^n$ and a matrix $M \in \mathbb{R}^{n\times n}$, we look at the problem of finding $\mathbf{u} \in \mathbb{R}^n$ such that

$$0 \leq \mathbf{u} \perp \mathbf{q} + M\mathbf{u} \geq 0. \tag{2}$$

The above problem is a Linear Complementarity Problem, denoted by LCP$(\mathbf{q}, M)$. There is a rich body of theory developed regarding LCPs. We will review here some basic results which will be needed in our subsequent discussions in Section 3. For more on LCP theory, [3] and [4] are good sources to refer to.

Given, $\mathbf{q} \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$, we denote the solution set of the LCP$(\mathbf{q}, M)$ by SOL$(\mathbf{q}, M)$. Naturally, the properties of the matrix M has a lot to do with questions of feasibility, existence, and uniqueness for a given LCP; this gives rise to the study of various classes of matrices in LCP theory. Here, we will talk about two major matrix classes which we will encounter in our discussions in this paper. For more on different matrix classes and their properties see for example [3] or [4].

Definition 2.1. (P-Matrix) We call a matrix $M \in \mathbb{R}^{n\times n}$ a P-Matrix, if the determinant of all of its principal sub-matrices are positive; that is, $\det(M_{\alpha\alpha}) > 0$ for all $\alpha \subseteq \{1, \ldots, n\}$.

Definition 2.2. (Copositive Matrix) Let $K \subseteq \mathbb{R}^n$ be a cone. We call a matrix $M \in \mathbb{R}^{n\times n}$ copositive on K if $\mathbf{x}^T M\mathbf{x} \geq 0$ for all $\mathbf{x} \in K$. Moreover, we call a matrix strictly copositive on K if $\mathbf{x}^T M\mathbf{x} > 0$ for all $\mathbf{x} \in K \setminus \{0\}$.

Remark 2.3. In some texts, a (strictly) copositive matrix is defined with the cone K explicitly taken to be the non-negative orthant.

$$K = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \geq 0\}.$$

Note that for a fixed matrix M, $\Phi(\mathbf{q}) = \mathrm{SOL}(\mathbf{q}, M)$ is a set-valued mapping. It is well known that in the case of M being a P-matrix, $\mathrm{LCP}(\mathbf{q}, M)$ has a unique solution for any $\mathbf{q} \in \mathbb{R}^n$ [7]; consequently, $\Phi(\mathbf{q})$ will be a singleton for any $\mathbf{q} \in \mathbb{R}^n$. We would like to say more on the properties of the solution map of an LCP; this leads us to discuss the notion of a *polyhedral multi-function.*

Let $\Phi : \mathbb{R}^n \mapsto 2^{\mathbb{R}^m}$ be a set-valued mapping (that is for any $\mathbf{x} \in \mathbb{R}^n$, $\Phi(\mathbf{x}) \subseteq \mathbb{R}^m$). Denote by $\Gamma(\Phi)$ the graph of Φ; that is,

$$\Gamma(\Phi) = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m \mid \mathbf{y} \in \Phi(\mathbf{x})\}.$$

We define a polyhedral multifunction as below [7].

Definition 2.4. (Polyhedral Multifunction) Let $\Phi : \mathbb{R}^n \mapsto 2^{\mathbb{R}^m}$ be a set-valued mapping. If there exists finitely many polyhedrons, $P^i \in \mathbb{R}^n \times \mathbb{R}^m$,

$$P^i = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m \mid B^i\mathbf{x} + A^i\mathbf{y} \leq \mathbf{b}^i\},$$

such that $\Gamma(\Phi) = \cup_{i=1}^N P^i$, we call Φ a polyhedral multifunction.

For convenience, we use the abbreviation PMF for a polyhedral multifunction from this point on. There is a special class of PMFs which is of special interest. Let $\Phi : \mathbb{R}^n \mapsto 2^{\mathbb{R}^m}$ be a PMF such that for any $\mathbf{x} \in \mathbb{R}^n$, $\Phi(\mathbf{x})$ is a singleton set; we call such a PMF a PMF with singleton property. The next result we are going to discuss brings to light the desirable properties of a PMF with singleton property; first we introduce the notion of a piecewise affine (linear) function [4].

Definition 2.5. Let $F : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a single-valued continuous mapping. We call F piece-wise affine (linear) if there exists finitely many affine (linear) functions $\{F^i\}_{i=1}^N$, (for some $N \in \mathbb{N}$), $F^i : \mathbb{R}^n \mapsto \mathbb{R}^m$, such that for all $\mathbf{x} \in \mathbb{R}^n$,

$$F(\mathbf{x}) \in \{F^1(\mathbf{x}), \cdots, F^N(\mathbf{x})\}.$$

From this point on, we will use the abbreviation PA (PL) for a piece-wise affine (linear) function.

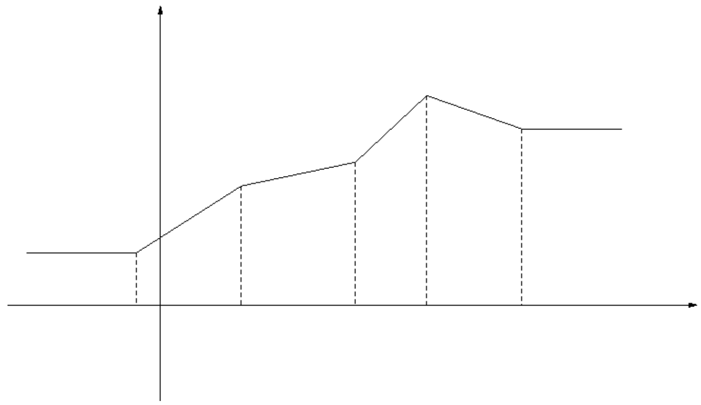

Figure 1: A piece-wise affine function

Theorem 2.6. *A PMF with singleton property is continuous and PA*[1]*.*

The next idea we need is that of a polyhedral subdivision.

Definition 2.7. Let Ξ be a finite collection of polyhedrons in $\mathbb{R}^n$, $\Xi = \{P^i\}_1^N$. We call Ξ a polyhedral subdivision of $\mathbb{R}^n$ if the following conditions hold [7]:

1. $\mathbb{R}^n = \cup_{i=1}^N P^i$.
2. Each P^i has dimension n.
3. The intersection any two of the polyhedrons is either empty or a proper common face.

There is a geometric way of looking at a PA function. Let us first consider a PA function on $\mathbb{R}$. In Figure 2.1, we see an example. We note that the function in Figure 2.1 induces a natural subdivision of $\mathbb{R}$ into intervals corresponding to each piece of the function. This idea can be extended into higher dimensions also; in that case, we will have a polyhedral subdivision of $\mathbb{R}^n$ induced by a PA function $F : \mathbb{R}^n \mapsto \mathbb{R}^m$.

Theorem 2.8. *[4] A continuous mapping $F : \mathbb{R}^n \mapsto \mathbb{R}^m$ is PA if and only if there exists a polyhedral subdivision $\Xi = \{P^i\}_1^N$ of $\mathbb{R}^n$ and a family of affine functions $\{F^i\}_{i=1}^N$ such that $F \equiv F^i$ on P^i.*

Theorem 2.8 can be used to prove the following useful result for PA functions [7].

Theorem 2.9. *Let $F : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a continuous PA function. Then, F is globally Lipschitz. That is, there exists a constant $K > 0$ such that $\left\|F(\mathbf{x}^1) - (\mathbf{x}^2)\right\| \leq K \left\|\mathbf{x}^1 - \mathbf{x}^2\right\|$ for all $\mathbf{x}^1$ and $\mathbf{x}^2$ in $\mathbb{R}^n$.*

The following is an easy consequence of the Theorems 2.6 and 2.9.

Corollary 2.10. *A PMF with singleton property is globally Lipschitz continuous.*

Now, we go back to study of LCP$(\mathbf{q}, M)$; one can show, by looking at the solution map of an LCP the following result [7].

[1] This result was proved first by M.S. Gowda.

Lemma 2.11. *For a fixed matrix $M \in \mathbb{R}^{n\times n}$, let $\Phi(\mathbf{q}) = \mathrm{SOL}(\mathbf{q}, M)$ (a set-valued mapping on $\mathbb{R}^n$). Then, $\Phi(\mathbf{q})$ is a PMF.*

Before moving further, we consider an LCP$(\mathbf{q}, M)$ where M is a P-Matrix; as we mentioned before, in this case, SOL$(\mathbf{q}, M)$ is a singleton for any $\mathbf{q} \in \mathbb{R}^n$; hence, by Lemma 2.11, $\mathbf{u}(\mathbf{q}) = \mathrm{SOL}(\mathbf{q}, M)$ will be a PMF with singleton property, and therefore is globally Lipschitz continuous. Hence, there exist a constant $K > 0$ such that for any $\mathbf{q}^1$, $\mathbf{q}^2 \in \mathbb{R}^n$,

$$\left\|\mathbf{u}(\mathbf{q}^1) - \mathbf{u}(\mathbf{q}^2)\right\| \leq K \left\|\mathbf{q}^1 - \mathbf{q}^2\right\|.$$

Finally, noting $\mathbf{u}(0) = 0$ we get that,

$$\|\mathbf{u}(\mathbf{q})\| \leq K \|\mathbf{q}\|, \quad \forall \mathbf{q} \in \mathbb{R}^n.$$

The special structure of the solution set of an LCP gives us more information in singleton case. For example, we have the following result [7].

Lemma 2.12. *Let $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{m\times m}$, be given such that $B\,\mathrm{SOL}(C\mathbf{x}, D)$ is a singleton for all $\mathbf{x} \in \mathbb{R}^n$. Then, $\Phi(\mathbf{x}) = B\,\mathrm{SOL}(C\mathbf{x}, D)$ is a continuous PL function and hence globally Lipschitz.*

2.2 Concepts form Stability Analysis of ODE Systems

In this sub-section, we recall some results from the stability theory for ODE systems[2]. Given, a function $f : \mathbb{R}^n \mapsto \mathbb{R}^n$, we consider the system,

$$\begin{cases} \dot{\mathbf{x}}(t) = f(\mathbf{x}), \\ \mathbf{x}(0) = \mathbf{x}^0. \end{cases} \tag{3}$$

Assuming f is Lipschitz is sufficient for well-posedness of the problem (3); that is, given an initial condition, $\mathbf{x}^0$, the corresponding state trajectory $\mathbf{x}(t, \mathbf{x}^0)$ will be unique given the right-hand side function f is Lipschitz continuous [1].

Recall that an equilibrium, $\mathbf{x}^e \in \mathbb{R}^n$ for (3) is a state such that $f(\mathbf{x}^e) = 0$. Also, if $\mathbf{x}^0 = \mathbf{x}^e$ then $\mathbf{x}(t, \mathbf{x}^0) = \mathbf{x}^e$ for all $t \geq 0$. Given an equilibrium $\mathbf{x}^e$ one may ask questions such as what will happen if we start with an initial condition very close to $\mathbf{x}^e$? If we start with an initial state sufficiently close to $\mathbf{x}^e$, will the trajectory stay close or level off at $\mathbf{x}^e$ in the long run? Such questions lead us to a formal study of the stability of ODE systems. In what follows we will study some fundamental results regarding stability analysis of ODE systems most relevant to the further developments in our subsequent discussions.

Definition 2.13. Consider a system,

$$\begin{cases} \dot{\mathbf{x}}(t) = f(\mathbf{x}), \\ \mathbf{x}(0) = \mathbf{x}^0, \end{cases} \tag{4}$$

and let $\mathbf{x}^e$ be an equilibrium state. We recall the following notions of stability [5].

1. We say $\mathbf{x}^e$ is stable in the sense of Lyapunov if for any $\epsilon > 0$ there exists a $\delta > 0$ such that
$$\left\|\mathbf{x}^0 - \mathbf{x}^e\right\| < \delta \quad \Longrightarrow \quad \left\|\mathbf{x}(t, \mathbf{x}^0) - \mathbf{x}^e\right\| < \epsilon, \quad \forall t \geq 0.$$

[2] We consider time-invariant systems only.

2. We say that $\mathbf{x}^e$ is asymptotically stable if it is stable in the sense of Lyapunov and for δ chosen above

$$\left\|\mathbf{x}^0 - \mathbf{x}^e\right\| < \delta \quad \Longrightarrow \quad \left\|\mathbf{x}(t, \mathbf{x}^0) - \mathbf{x}^e\right\| \to 0, \quad \text{as } t \to \infty.$$

3. We say that $\mathbf{x}^e$ is exponentially stable if there exists $\delta > 0$, $k > 0$, and $\mu > 0$ such that

$$\left\|\mathbf{x}^0 - \mathbf{x}^e\right\| < \delta \quad \Longrightarrow \quad \left\|\mathbf{x}(t, \mathbf{x}^0) - \mathbf{x}^e\right\| \le k e^{-\mu t} \left\|\mathbf{x}^0 - \mathbf{x}^e\right\|, \quad \forall t \ge 0.$$

It is clear that exponential stability implies asymptotic stability and asymptotic stability implies Lyapunov stability; these implications cannot be reversed in general. However, in the special case of a linear system, where the right-hand side function f is linear ($f(\mathbf{x}) = A\mathbf{x}$ for a matrix A), it is well known that the notions of asymptotic stability and exponential stability are equivalent [7].

Note that when considering an equilibrium state $\mathbf{x}^e$ of (4), we can, without loss of generality, assume $\mathbf{x}^e = 0$; suppose we have an equilibrium $\mathbf{x}^e \neq 0$, it is trivial matter to do a change of coordinates so that $\mathbf{x}^e$ is shifted to origin. To be more precise, given $\mathbf{x}^e \neq 0$ use the change of variable

$$\mathbf{z} = \mathbf{x} - \mathbf{x}^e.$$

Then, we have

$$\dot{\mathbf{z}} = \dot{\mathbf{x}} = f(\mathbf{x}) = f(\mathbf{z} + \mathbf{x}^e) := \bar{f}(\mathbf{z}).$$

Now note that $\mathbf{z} = 0$ is an equilibrium for the system $\dot{\mathbf{z}} = \bar{f}(\mathbf{z})$. Next, we look at the following simple result regarding asymptotic stability when the right-hand side function f of the system (3) is a positively homogeneous function; recall that a function $f(\mathbf{x})$ is positively homogeneous if $f(\tau\mathbf{x}) = \tau f(\mathbf{x})$ for any $\tau \ge 0$.

Lemma 2.14. *Suppose the right-hand side function f in (3) is positively homogeneous. Let $\mathbf{x}^e = 0$ be an equilibrium of (3); if $\mathbf{x}^e$ is asymptotically stable then* $\mathrm{Ker}\{\mathrm{f}\} = \{0\}$ *(where* $\mathrm{Ker}\{\mathrm{f}\} = \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) = 0\}$*).*

Proof. Suppose to the contrary that there exits $0 \neq \bar{\mathbf{x}} \in \mathrm{Ker}\{\mathrm{f}\}$. Then, for any $\epsilon > 0$, $f(\epsilon\bar{\mathbf{x}}) = \epsilon f(\bar{\mathbf{x}}) = 0$. So $\epsilon\bar{\mathbf{x}}$ is also an equilibrium for any $\epsilon > 0$.

Now for any $\delta > 0$, we can let $\mathbf{z}^\delta = \frac{\delta}{2}\frac{\bar{\mathbf{x}}}{\|\bar{\mathbf{x}}\|}$, so that $\left\|\mathbf{z}^\delta - \mathbf{x}^e\right\| = \left\|\mathbf{z}^\delta\right\| < \delta$. Also, we have by the previous argument that $\mathbf{z}^\delta$ is an equilibrium state. Subsequently,

$$\left\|\mathbf{x}(t, \mathbf{z}^\delta) - \mathbf{x}^e\right\| = \left\|\mathbf{z}^\delta\right\| \neq 0 \quad \forall t \ge 0.$$

However, this contradicts the asymptotic stability of $\mathbf{x}^e = 0$ (the convergence property is violated). Therefore, it follows that $\mathrm{Ker}\{\mathrm{f}\} = \{0\}$. ☐

For a linear system,

$$\begin{cases} \dot{\mathbf{x}}(t) = A\mathbf{x}, \\ \mathbf{x}(0) = \mathbf{x}^0, \end{cases} \tag{5}$$

where $A \in \mathbb{R}^{n \times n}$, we can use the above result to conclude that a necessary condition for $\mathbf{x}^e = 0$ to be asymptotically stable is that[3] $\mathrm{Rank}(A) = n$.

[3] This rank condition on A is trivial to show using the more specialized results for linear ODE systems, where we consider the eigen-values of A.

Now, we consider more sophisticated results. The stability theory of ODE systems is a vast area of study. There are many specialized results, for instance, for the case of linear systems which we will not discuss here; instead, we stay focused on results more relevant to our discussion. The following well known result provides a sufficient condition for Lyapunov stability of a non-linear system [7].

Theorem 2.15. *Consider the following non-linear system*

$$\begin{cases} \dot{\mathbf{x}}(t) = f(\mathbf{x}), \\ \mathbf{x}(0) = \mathbf{x}^0, \end{cases} \tag{6}$$

with a Lipschitz $f : \mathbb{R}^n \mapsto \mathbb{R}^n$. *Let* $\mathbf{x}^e = 0$ *be an equilibrium state. Suppose there exists a neighborhood* $\mathcal{N}$ *of* $\mathbf{x}^e$ *and a continuously differentiable function* $V : \mathbb{R}^n \mapsto \mathbb{R}$ *such that*

$$\begin{array}{ll} V(\mathbf{x}^e) = 0, \;\; \textit{and } V(\mathbf{x}) > 0 & \forall \mathbf{x} \in \mathcal{N} \setminus \{0\}, \\ \dot{V}(\mathbf{x}(t, \mathbf{x}^0)) \leq 0, & \textit{for } \mathbf{x}(t, \mathbf{x}^0) \in \mathcal{N}. \end{array}$$

Then, $\mathbf{x}^e = 0$ *is stable in the sense of Lyapunov. Moreover, if* $\dot{V}(\mathbf{x}(t, \mathbf{x}^0)) < 0$ *, for* $\mathbf{x}(t, \mathbf{x}^0) \in \mathcal{N}$ *in above conditions, then* $\mathbf{x}^e = 0$ *is asymptotically stable.*

The above result, called Lyapunov's direct method, is an elegant theoretical construct. However, in practice, it is not so easy to apply Theorem 2.15 for the very simple reason that coming up with a function V satisfying the conditions of the theorem is not always an easy task. However, the result is still fundamental in the stability theory of non-linear systems. Note that finding V such that the above theorem gives asymptotic stability is even more difficult. Hence, one may ask, under what conditions the requirements on V can be relaxed in such a way that we can still get asymptotic stability. The answer to this question is eventually given by LaSalle's Theorem. Before discussing LaSalle's result, however, we need some technical preparations.

Definition 2.16. (Positive Limit Set [7]) Let an initial condition $\mathbf{x}^0 \in \mathbb{R}^n$ of the time invariant non-linear system

$$\begin{cases} \dot{\mathbf{x}}(t) = f(\mathbf{x}), \\ \mathbf{x}(0) = \mathbf{x}^0, \end{cases} \tag{7}$$

be given. If there exists a sequence $\{t^k\}_{t^k \geq 0}$, such that $\lim_{k \to \infty} \mathbf{x}(t, \mathbf{x}^0)$ exists. Then, we denote $\mathbf{x}^\infty := \lim_{k \to \infty} \mathbf{x}(t, \mathbf{x}^0)$, and call $\mathbf{x}^\infty$ a positive limit point of $\mathbf{x}^0$. We then call the set of all positive limit points of $\mathbf{x}^0$ its positive limit set and denote this set by $\Omega(\mathbf{x}^0)$.

Definition 2.17. (Positive Invariance [7]) Consider the ODE system (7) from the last definition. A set $M \subset \mathbb{R}^n$ is a positively invariant set (with respect to (7)) if the following holds:

$$\mathbf{x}^0 \in M \quad \Longrightarrow \quad \mathbf{x}(t, \mathbf{x}^0) \in M, \, \forall t \geq 0.$$

The following is a standard technical result regarding the positive limit set [7]. Again consider the problem (7).

Lemma 2.18. *Let* $\mathbf{x}(t, \mathbf{x}^0)$ *be a trajectory of (7) with initial state* $\mathbf{x}(0) = \mathbf{x}^0$. *Suppose there exists a compact set* $\mathcal{N}$ *such that* $\mathbf{x}(t, \mathbf{x}^0) \in \mathcal{N}$ *for all* $t \geq 0$. *Then,* $\Omega(\mathbf{x}^0)$ *is non-empty, compact, connected, and positively invariant.*

The above technical Lemma is used in the proof of the following well known stability result [7].

Theorem 2.19. *Let $\mathbf{x}^e = 0$ be an equilibrium of (7); assume the right-hand side function f is Lipschitz on a neighborhood $\mathcal{N}$ of $\mathbf{x}^0$. Let V be a C^1 function satisfying the following conditions.*

- $V(0) = 0$, *and* $V(x) > 0$, *for all* $0 \neq \mathbf{x} \in \mathcal{N}$.
- $\dot{V}(\mathbf{x}(t, \mathbf{x}^0)) \leq 0$, *for* $\mathbf{x}(t, \mathbf{x}^0) \in \mathcal{N}$.

Then, we have the following

1. *There exists neighborhood* $\tilde{\mathcal{N}}$ *of* $\mathbf{x}^e = 0$ *such that for all* $\mathbf{x}^0 \in \tilde{\mathcal{N}}$, *we have* $\mathbf{x}(t, \mathbf{x}^0) \in \mathcal{N}$, $\forall t \geq 0$.
2. *For all* $\mathbf{x}^0 \in \tilde{\mathcal{N}}$, *there exists a constant* $c = c(\mathbf{x}^0)$ *such that*
   - $V(\mathbf{x}^\infty) = c(\mathbf{x}^0), \quad \forall \mathbf{x}^\infty \in \Omega(\mathbf{x}^0)$
   - $\dot{V}(\mathbf{x}(t, \mathbf{x}^\infty)) = 0$ *for all* $t \geq 0$, *for any* $\mathbf{x}^\infty \in \Omega(\mathbf{x}^0)$.
3. *If in addition to the above conditions on* V *we also have on a compact neighborhood* $\hat{\mathcal{N}}$ *containing* $\mathcal{N}$,
$$\dot{V}(\mathbf{x}(t, \mathbf{x}^\infty)) = 0 \quad \Longrightarrow \quad \mathbf{x}^0 = 0. \tag{8}$$
   Then $\mathbf{x}^e = 0$ *is asymptotically stable.*

The third statement of the above Theorem is the well known LaSalle's result. Note that in Theorem 2.15, we needed a Lyapunov function V which is positive on a compact neighborhood $\mathcal{N}$ of $\mathbf{x}^e = 0$, and $\dot{V}(\mathbf{x}(t, \mathbf{x}^0)) < 0$ for $\mathbf{x}(t, \mathbf{x}^0) \in \mathcal{N}$, to get asymptotic stability. On the other hand, in Theorem 2.19, we only require $\dot{V}(\mathbf{x}(t, \mathbf{x}^0)) \leq 0$ for $\mathbf{x}(t, \mathbf{x}^0) \in \mathcal{N}$ which is a less stringent condition. However, we had to compensate for this weaker assumption by the added hypotheses we saw in the Theorem 2.19. As we will see in Section 3, an extension of LaSalle's result can be proved to test for asymptotic stability in the case of an LCS.

3 Stability of Linear Complementarity Systems

Given constant matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, and $D \in \mathbb{R}^{m \times m}$, consider the LCS$(A, B, C, D)$ which is the problem of finding the state trajectory $\mathbf{x} = \mathbf{x}(t) \in \mathbb{R}^n$ and control $\mathbf{u} = \mathbf{u}(t) \in \mathbb{R}^m$ such that

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \\ 0 \leq \mathbf{u} \perp C\mathbf{x} + D\mathbf{u} \geq 0, \\ \mathbf{x}(0) = \mathbf{x}^0. \end{cases} \tag{9}$$

Suppose we have that $B\,\mathrm{SOL}(C\mathbf{x}, D)$ is a singleton[4] for any $\mathbf{x} \in \mathbb{R}^n$. Then (9) is the same as

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + B\,\mathrm{SOL}(C\mathbf{x}, D), \\ \mathbf{x}(0) = \mathbf{x}^0. \end{cases} \tag{10}$$

[4] More precisely, we assume that For any $\mathbf{x} \in \mathbb{R}^n$, $B\,\mathrm{SOL}(C\mathbf{x}, D)$ is non-empty and is a singleton set.

Also, using singleton property of $B\,\mathrm{SOL}(C\mathbf{x}, D)$, we know the mapping $\mathbf{x} \mapsto B\,\mathrm{SOL}(C\mathbf{x}, D)$ is globally Lipschitz (use Lemma 2.12). Therefore the right-hand side function in (10) is globally Lipschitz continuous; hence, we know that there exists a unique C^1 state trajectory for any given $\mathbf{x}^0$. In fact we can be more precise by stating the following result [2].

Theorem 3.1. *Consider the LCS (9). Then, the following are equivalent.*

1. *For any $\mathbf{x}^0 \in \mathbb{R}^n$, the LCS (9) has a unique C^1 state trajectory, $\mathbf{x}(t, \mathbf{x}^0)$, for all t.*
2. *For any $\mathbf{x}^0 \in \mathbb{R}^n$, the set $B\,\mathrm{SOL}(C\mathbf{x}(t, \mathbf{x}^0), D)$ is a singleton for every t.*

Proof. (1) $\Rightarrow$ (2): Let $\mathbf{x}^0 \in \mathbb{R}^n$ be any initial state. Fix a time t^* and let $\mathbf{u}^1$ and $\mathbf{u}^2$ be in $\mathrm{SOL}(C\mathbf{x}(t^*, \mathbf{x}^0), D)$. We know by (1) that $\mathbf{x}(t^*, \mathbf{x}^0)$ is uniquely defined. Then, we have $B\mathbf{u}^1 = \dot{\mathbf{x}}(t^*, \mathbf{x}^0) - A\mathbf{x}(t^*, \mathbf{x}^0) = B\mathbf{u}^2$; hence, we have (2).

(2) $\Rightarrow$ (1): Consider $t = 0$; then, (2) says that For any $\mathbf{x}^0 \in \mathbb{R}^n$, the set $B\,\mathrm{SOL}(C\mathbf{x}^0, D)$ is a singleton. Then, (1) follows by our earlier discussions above. □

As noted by authors in [2] assuming $B\,\mathrm{SOL}(C\mathbf{x}, D)$ is a singleton for all $\mathbf{x} \in \mathbb{R}^n$, is not actually a very restrictive assumption and there are many interesting cases where this condition is satisfied; under this singleton assumption, instead of discussing the LCS (9), we can instead examine the problem (10), which is repeated below for convenience.

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + B\,\mathrm{SOL}(C\mathbf{x}, D) \\ \mathbf{x}(0) = \mathbf{x}^0. \end{cases} \tag{11}$$

Then, to discuss the stability of LCS (9), we can examine the above problem. Note that at this point we have a ODE system, with a Lipschitz continuous right-hand side, whose stability can be put in terms of stability ideas discussed in Section 2.2.

We can get a result regarding asymptotic stability of $\mathbf{x}^e = 0$ as follows. Consider problem (11); first recall that if $\mathbf{u} \in \mathrm{SOL}(C\mathbf{x}, D)$ then $\epsilon\mathbf{u} \in \mathrm{SOL}(C(\epsilon\mathbf{x}), D)$ for any $\epsilon > 0$. Now, we want to show for all $\epsilon > 0$, $B\,\mathrm{SOL}(C(\epsilon\mathbf{x}), D) = \epsilon B\,\mathrm{SOL}(C\mathbf{x}, D)$ for any fixed $\mathbf{x}$. Let $\mathbf{u}^0$ be some element in $\mathrm{SOL}(C\mathbf{x}, D)$; By singleton property of $B\,\mathrm{SOL}(C\mathbf{x}, D)$ we know $B\mathbf{u}^0 = B\mathbf{u}$, for any $\mathbf{u} \in \mathrm{SOL}(C\mathbf{x}, D)$; that is $B\mathbf{u}^0 = B\,\mathrm{SOL}(C\mathbf{x}, D)$. Next, note that for a $\mathbf{u}^1 \in \mathrm{SOL}(C(\epsilon\mathbf{x}), D)$, we know $\frac{1}{\epsilon}\mathbf{u}^1 \in \mathrm{SOL}(C\mathbf{x}, D)$. Therefore, $B\mathbf{u}^1 = \epsilon B(\frac{1}{\epsilon}\mathbf{u}^1) = \epsilon B\mathbf{u}^0$. Therefore,

$$B\,\mathrm{SOL}(C(\epsilon\mathbf{x}), D) = B\mathbf{u}^1 = \epsilon B\mathbf{u}^0 = \epsilon B\,\mathrm{SOL}(C\mathbf{x}, D).$$

Subsequently, we note that the function $f(\mathbf{x}) = A\mathbf{x} + B\,\mathrm{SOL}(C\mathbf{x}, D)$ is positively homogeneous in $\mathbf{x}$. Then, using Lemma 2.14 we have the following result.

Corollary 3.2. *Consider the equilibrium $\mathbf{x}^e = 0$ of (11); if $\mathbf{x}^e$ is asymptotically stable then* $\mathrm{Ker}\{\mathrm{A}\mathbf{x} + \mathrm{B\,SOL(C}\mathbf{x}, \mathrm{D)}\} = \{0\}$.

Proof. By the above argument, the right-hand side function $f(\mathbf{x}) = A\mathbf{x} + B\,\mathrm{SOL}(C\mathbf{x}, D)$ is positively homogeneous. Hence, the result follows by applying Lemma 2.14. □

Using the above Corollary, we can get the following result.

Corollary 3.3. *Consider the problem (11), where we assume $B\,\mathrm{SOL}(C\mathbf{x}, D)$ is singleton for all $\mathbf{x} \in \mathbb{R}^n$. If the following system is solvable*

$$\begin{cases} A\mathbf{x} = 0, \\ C\mathbf{x} \gneq 0. \end{cases} \tag{12}$$

Then the equilibrium $\mathbf{x}^e = 0$ is not asymptotically stable.

Proof. Suppose $\bar{\mathbf{x}}$ solves (12). First note that $C\bar{\mathbf{x}} \gneq 0$ implies $\bar{\mathbf{x}} \neq 0$. Next, since $C\bar{\mathbf{x}} \geq 0$, we know that $\mathbf{u}^0 = 0$ is a solution to $\mathrm{LCP}(C\bar{\mathbf{x}}, D)$; then, by the singleton property of $B\,\mathrm{SOL}(C\bar{\mathbf{x}}, D)$, we know $B\,\mathrm{SOL}(C\bar{\mathbf{x}}, D) = B\mathbf{u}^0 = 0$. Therefore, $A\bar{\mathbf{x}} + B\,\mathrm{SOL}(C\bar{\mathbf{x}}, D) = 0$, and hence $\mathrm{Ker}\{\mathrm{A}\bar{\mathbf{x}} + \mathrm{B\,SOL}(\mathrm{C}\bar{\mathbf{x}}, \mathrm{D})\} \neq \{0\}$. Therefore, by Corollary 3.2, we know that $\mathbf{x}^e = 0$ is not asymptotically stable. □

3.1 Asymptotic Stability for $\mathbf{x}^e = 0$

In the sequel, our goal is to provide sufficient condition for asymptotic stability of an equilibrium $\mathbf{x}^e = 0$ of the LCS (9). Authors in [2] achieved this through an extension to LaSalle's Theorem (see Theorem 2.19). Here we consider the case where D is a P-matrix in which case, $\mathrm{SOL}(C\mathbf{x}, D)$ is a singleton for all $\mathbf{x} \in \mathbb{R}^n$; that is, given any $\mathbf{x} \in \mathbb{R}^n$ we have a corresponding unique $\mathbf{u}(\mathbf{x})$. Moreover, by the Lipschitz property of the solution map $\mathbf{u}(\mathbf{x}) = \mathrm{SOL}(C\mathbf{x}, D)$ (see results in Section 2.1), we know there exists $c_D > 0$ such that for any $\mathbf{x} \in \mathbb{R}^n$,

$$\|\mathbf{u}(\mathbf{x})\| \leq c_D \|\mathbf{x}\|. \tag{13}$$

In what follows, the graph $\Gamma(\mathbf{u})$ of the solution map, $\mathbf{u}(\mathbf{x}) = \mathrm{SOL}(C\mathbf{x}, D)$,

$$\Gamma(\mathbf{u}) = \{(\mathbf{x}, \mathbf{u}(\mathbf{x})) \in \mathbb{R}^n \times \mathbb{R}^m \mid \mathbf{x} \in \mathbb{R}^n\},$$

will be of special importance. Note that $\Gamma(\mathbf{u})$ is a closed cone (not necessarily convex).

We also consider the directional derivative of $\mathbf{u}(\mathbf{x})$ in direction $\mathbf{d}$,

$$\mathbf{u}'(\mathbf{x}; \mathbf{d}) = \lim_{\tau \downarrow 0} \frac{\mathbf{u}(\mathbf{x} + \tau\mathbf{d}) - \mathbf{u}(\mathbf{x})}{\tau}.$$

Then, following the discussion in [2], we define the map, $\mathrm{SOL}'_{\mathrm{LCS}} : \mathbb{R}^n \mapsto \mathbb{R}^m \times \mathbb{R}^m$ by

$$\mathrm{SOL}'_{\mathrm{LCS}}(\mathbf{x}) = \begin{bmatrix} \mathbf{u}(\mathbf{x}) \\ \mathbf{u}'(\mathbf{x}; \mathbf{dx}) \end{bmatrix},$$

where $\mathbf{dx} = A\mathbf{x} + B\mathbf{u}(\mathbf{x})$. Again, we look at the graph, $\Gamma(\mathrm{SOL}'_{\mathrm{LCS}})$. Note that $\Gamma(\mathrm{SOL}'_{\mathrm{LCS}})$ is a cone; however, unlike $\Gamma(\mathbf{u})$, it is not in general closed, for $\mathbf{u}'(\mathbf{x}; \mathbf{d})$ may be discontinuous in $\mathbf{x}$.

Now, with the assumption that D is a P-matrix, we know that for any $\mathbf{x} \in \mathbb{R}^n$, there is a unique $\mathbf{u}(\mathbf{x}) \in \mathbb{R}^m$; thus, LCS (9) becomes

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}(\mathbf{x}) \\ \mathbf{x}(0) = \mathbf{x}^0. \end{cases} \tag{14}$$

Note that the right-hand side function $f(\mathbf{x}) = A\mathbf{x}+B\mathbf{u}(\mathbf{x})$ is PL and $f(\mathbf{x}) = 0$ at the origin. We follow the approach in [2] where existence of a symmetric matrix $M \in \mathbb{R}^{(n+m)\times(n+m)}$

$$M = \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix} \tag{15}$$

which is strictly copositive on the cone $\Gamma(\mathbf{u})$ is postulated. We immediately note the following.

Lemma 3.4. *There exists a constant $c_M > 0$ such that $\mathbf{y}^T M\mathbf{y} \geq c_M \mathbf{y}^T \mathbf{y}$ for all $\mathbf{y} \in \Gamma(\mathbf{u})$.*

Proof. By strict copositivity of M on $\Gamma(\mathbf{u})$ we have

$$\mathbf{y}^T M\mathbf{y} > 0 \quad \forall \mathbf{y} \in \Gamma(\mathbf{u}) \setminus \{0\}.$$

Now recall that $\Gamma(\mathbf{u})$ is closed also; thus, $f(\mathbf{y}) = \mathbf{y}^T M\mathbf{y} > 0$ is continuous and bounded from below on the compact set $K = \{\mathbf{y} \in \Gamma(\mathbf{u}) \mid \|\mathbf{y}\| = 1\}$. Therefore, there is a $\mathbf{y}_0 \in K$ such that $f(\mathbf{y}) \geq f(\mathbf{y}_0)$ for all $\mathbf{y} \in K$. Let $c_M = f(\mathbf{y}_0) > 0$ and note that

$$f(\mathbf{y}) = \mathbf{y}^T M\mathbf{y} \geq c_M \mathbf{y}^T \mathbf{y}, \quad \forall \mathbf{y} \in \Gamma(\mathbf{u}).$$

□

With this preparations, we define [2]

$$V(\mathbf{x},\mathbf{u}) = \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix}^T \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix}. \tag{16}$$

Consider also the composite function

$$\hat{V}(\mathbf{x}) := V(\mathbf{x},\mathbf{u}(\mathbf{x})) = \mathbf{x}^T P\mathbf{x} + 2\mathbf{x}^T Q\mathbf{u}(\mathbf{x}) + \mathbf{u}(\mathbf{x})^T R\mathbf{u}(\mathbf{x}).$$

Note that $\hat{V}$ is locally Lipschitz and has directional derivatives [2],

$$\hat{V}'(\mathbf{x};\mathbf{v}) = 2\mathbf{x}^T P\mathbf{v} + 2\mathbf{v}^T Q\mathbf{u}(\mathbf{x}) + 2\mathbf{x}^T Q\mathbf{u}'(\mathbf{x};\mathbf{v}) + 2\mathbf{u}(\mathbf{x})^T R\mathbf{u}'(\mathbf{x};\mathbf{v}).$$

Now consider trajectories of the LCS (14), $(\mathbf{x}(t,\mathbf{x}^0), \mathbf{u}(\mathbf{x}(t,\mathbf{x}^0)))$. For notational convenience, we denote $\mathbf{u}(\mathbf{x}(t,\mathbf{x}^0)) = \mathbf{u}(t,\mathbf{x}^0)$. Then, define the function

$$\phi_{\mathbf{x}^0}(t) := \hat{V}(\mathbf{x}(t,\mathbf{x}^0)), \quad \forall t \geq 0.$$

Next, we use chain rule for directional derivatives to get the one sided derivative of $\phi_{\mathbf{x}^0}(t)$ [2],

$$\begin{aligned}\phi'_{\mathbf{x}^0}(t+) &= \lim_{\tau\downarrow 0} \frac{\phi_{\mathbf{x}^0}(t+\tau) - \phi_{\mathbf{x}^0}(t)}{\tau} \\ &= \hat{V}'(\mathbf{x}(t,\mathbf{x}^0); \dot{\mathbf{x}}(t,\mathbf{x}^0)) \\ &= 2\mathbf{x}(t)^T P\dot{\mathbf{x}}(t) + 2\dot{\mathbf{x}}(t)^T Q\mathbf{u}(\mathbf{x}(t)) + 2\mathbf{x}(t)^T Q\mathbf{u}'(\mathbf{x}(t);\dot{\mathbf{x}}(t)) + 2\mathbf{u}(t)^T R\mathbf{u}'(\mathbf{x}(t);\dot{\mathbf{x}}(t)),\end{aligned}$$

where in the last equality we have suppressed the dependence on $\mathbf{x}^0$ to make the notation clearer. Now, let

$$\mathbf{v}(t,\mathbf{x}^0) = \mathbf{u}'(\mathbf{x}(t,\mathbf{x}^0); \dot{\mathbf{x}}(t,\mathbf{x}^0)),$$

and recall that from LCS (14) we have

$$\dot{\mathbf{x}}(t,\mathbf{x}^0) = A\mathbf{x}(t,\mathbf{x}^0) + B\mathbf{u}(t,\mathbf{x}^0).$$

Substituting the expression for $\dot{\mathbf{x}}(t,\mathbf{x}^0)$ into the expression for $\phi'_{\mathbf{x}^0}(t+)$ will result in a rather long expression. To have a more compact form for $\phi'_{\mathbf{x}^0}(t+)$, we develop the following notation (recalling symmetry of P and R) [2]. Let $\mathbf{z}(t,\mathbf{x}^0) \in \Gamma(\mathrm{SOL}'_{\mathrm{LCS}})$ be given by

$$\mathbf{z}(t,\mathbf{x}^0) = \begin{bmatrix} \mathbf{x}(t,\mathbf{x}^0) \\ \mathbf{u}(t,\mathbf{x}^0) \\ \mathbf{v}(t,\mathbf{x}^0) \end{bmatrix},$$

and define the matrix

$$N = \begin{bmatrix} A^TP + PA & PB + A^TQ & Q \\ B^TP + Q^TA & Q^TB + B^TQ & R \\ Q^T & R & 0 \end{bmatrix}.$$

Then, we have that

$$\phi'_{\mathbf{x}^0}(t+) = \mathbf{z}(t,\mathbf{x}^0)^T N \mathbf{z}(t,\mathbf{x}^0). \tag{17}$$

Now using the notation developed so far, we set out to derive sufficient conditions for stability of $\mathbf{x}^e = 0$.

In what follows we assume the matrix D in LCS(A,B,C,D) is a P-matrix, and assume there exists a symmetric matrix $M \in \mathbb{R}^{(n+m)\times(n+m)}$

$$M = \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix} \tag{18}$$

which is strictly copositive on the cone $\Gamma(\mathbf{u})$.

In proving the first stability result, we will need the following theorem on differentiability of locally Lipschitz functions [9].

Theorem 3.5. *(Radamacher) If a function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ is locally Lipschitz on an open set Ω, then it is differentiable almost everywhere on Ω.*

The first stability result we discuss gives sufficient conditions for linear bounded stability of $\mathbf{x}^e = 0$ (the following result is part (a) of Theorem 3.1 in [2]).

Theorem 3.6. *Assume that $-N$ is copositive on $\Gamma(\mathrm{SOL}'_{\mathrm{LCS}})$. Then, $\mathbf{x}^e = 0$ is linearly bounded stable. That is, there exists a constant $\rho > 0$ such that for any $\mathbf{x}^0 \in \mathbb{R}^n$,*

$$\left\|\mathbf{x}(t,\mathbf{x}^0)\right\| \le \rho \left\|\mathbf{x}^0\right\|, \quad \forall t \ge 0.$$

Proof. Let $\mathbf{x}^0 \in \mathbb{R}^n$ be arbitrary, and let $\mathbf{u}^0 := \mathbf{u}(\mathbf{x}^0)$. Note that for all $t \ge 0$, $\phi_{\mathbf{x}^0}(t) = \hat{V}(\mathbf{x}(t,\mathbf{x}^0))$ is locally Lipschitz continuous and hence, by Theorem 3.5 is differentiable almost everywhere on $(0,\infty)$. Therefore for almost all $t \ge 0$, $\phi'_{\mathbf{x}^0}(t)$ exists and equals $\phi'_{\mathbf{x}^0}(t+)$. Also, recall that $\phi'_{\mathbf{x}^0}(t+) = \mathbf{z}(t, \mathbf{x}^0)^T N \mathbf{z}(t, \mathbf{x}^0)$. Thus, by copositivity of $-N$ on $\Gamma(\mathrm{SOL}'_{\mathrm{LCS}})$ we know

$$\phi'_{\mathbf{x}^0}(t+) \leq 0, \quad \forall t \geq 0.$$

Next, note that for all $t \geq 0$,

$$\phi_{\mathbf{x}^0}(t) = \phi_{\mathbf{x}^0}(0) + \int_0^t \phi'_{\mathbf{x}^0}(s+)\, ds \leq \phi_{\mathbf{x}^0}(0) = V(\mathbf{x}^0, \mathbf{u}^0). \tag{19}$$

Moreover, we have (using Cauchy-Schwartz inequality and (13))

$$\begin{aligned}
V(\mathbf{x}^0, \mathbf{u}^0) &= \begin{bmatrix} \mathbf{x}^0 \\ \mathbf{u}^0 \end{bmatrix}^T \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix} \begin{bmatrix} \mathbf{x}^0 \\ \mathbf{u}^0 \end{bmatrix}. \\
&= (\mathbf{x}^0)^T P \mathbf{x}^0 + (\mathbf{x}^0)^T Q \mathbf{u}^0 + (\mathbf{u}^0)^T Q^T \mathbf{x}^0 + (\mathbf{u}^0)^T R \mathbf{u}^0 \\
&\leq \|P\| \left\|\mathbf{x}^0\right\|^2 + 2\|Q\| \left\|\mathbf{x}^0\right\| \left\|\mathbf{u}^0\right\| + \|R\| \left\|\mathbf{u}^0\right\|^2 \\
&\leq \|P\| \left\|\mathbf{x}^0\right\|^2 + 2c_D \|Q\| \left\|\mathbf{x}^0\right\|^2 + c_D^2 \|R\| \left\|\mathbf{x}^0\right\|^2 \\
&= \left(\|P\| + 2c_D \|Q\| + c_D^2 \|R\|\right) \left\|\mathbf{x}^0\right\|^2.
\end{aligned}$$

Thus, we see that there is a constant, $\rho_M > 0$ independent of $\mathbf{x}^0$ (here $\rho_M = \|P\| + 2c_D \|Q\| + c_D^2 \|R\|$) such that

$$V(\mathbf{x}^0, \mathbf{u}^0) \leq \rho_M \left\|\mathbf{x}^0\right\|^2. \tag{20}$$

Thus, by (19) and (20) we have

$$\phi_{\mathbf{x}^0}(t) \leq \rho_M \left\|\mathbf{x}^0\right\|^2. \tag{21}$$

Next, we note that by strict copositivity of M on $\Gamma(\mathbf{u})$ there exists a constant $c_M > 0$ such that (recall Lemma 3.4),

$$\begin{aligned}
V(\mathbf{x}(t, \mathbf{x}^0), \mathbf{u}(t, \mathbf{x}^0)) &= \begin{bmatrix} \mathbf{x}(t, \mathbf{x}^0) \\ \mathbf{u}(t, \mathbf{x}^0) \end{bmatrix}^T \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix} \begin{bmatrix} \mathbf{x}(t, \mathbf{x}^0) \\ \mathbf{u}(t, \mathbf{x}^0) \end{bmatrix} \\
&\geq c_M \left\| \begin{bmatrix} \mathbf{x}(t, \mathbf{x}^0) \\ \mathbf{u}(t, \mathbf{x}^0) \end{bmatrix} \right\|^2 \\
&\geq c_M \left\|\mathbf{x}(t, \mathbf{x}^0)\right\|^2.
\end{aligned}$$

Hence,

$$\phi_{\mathbf{x}^0}(t) = V(\mathbf{x}(t, \mathbf{x}^0), \mathbf{u}(t, \mathbf{x}^0)) \geq c_M \left\|\mathbf{x}(t, \mathbf{x}^0)\right\|^2. \tag{22}$$

Then, combining (21) and (22) we have

$$\left\|\mathbf{x}(t, \mathbf{x}^0)\right\| \leq \rho \left\|\mathbf{x}^0\right\|, \tag{23}$$

with $\rho = \sqrt{\frac{\rho_M}{c_M}}$; this completes the proof. □

Next, we will follow the authors in [2] by considering the following extension of LaSalle's Theorem; the result below is part (c) of Theorem 3.1 in [2].

Theorem 3.7. *Assume $-N$ is copositive on $\Gamma(\mathrm{SOL}'_{\mathrm{LCS}})$ and we have*

$$\left[\mathbf{z}(t,\xi)^T N \mathbf{z}(t,\xi) = 0, \quad \forall t \geq 0\right] \Longrightarrow \xi = 0.$$

Then, the equilibrium $\mathbf{x}^e = 0$ is asymptotically stable.

To prove the above Theorem, we need the following technical lemma (proof of which follows closely that of Proposition 3.2 in [2]).

Lemma 3.8. *Suppose $-N$ is copositive on $\Gamma(\mathrm{SOL}'_{\mathrm{LCS}})$. Then, for any $\mathbf{x}^0 \in \mathbb{R}^n$ the following statements hold.*

1. *$\Omega(\mathbf{x}^0) \neq \emptyset$.*
2. *$\Omega(\mathbf{x}^0)$ is positively invariant.*
3. *There exists a constant $\kappa_{\mathbf{x}^0}$ such that*

$$V(\mathbf{x}^\infty, \mathbf{u}(\mathbf{x}^\infty)) = \kappa_{\mathbf{x}^0}, \quad \forall \mathbf{x}^\infty \in \Omega(\mathbf{x}^0).$$

4. *$\phi'_{\mathbf{x}^\infty} \equiv 0$ for every $\mathbf{x}^\infty \in \Omega(\mathbf{x}^0)$.*

Proof. Let $\mathbf{x}^0 \in \mathbb{R}^n$ be arbitrary.

(1): This actually follows from Theorem 3.6; we know that there is a constant $\rho > 0$ such that $\mathbf{x}(t,\mathbf{x}^0) \in \bar{B}(\mathbf{x}^0, \rho \left\|\mathbf{x}^0\right\|)$ for all $t \geq 0$. Take a sequence $\{t^k\}_{t^k \geq 0}$, such that $t^k \uparrow \infty$. Let $\mathbf{x}^k = \mathbf{x}(t^k, \mathbf{x}^0)$, for $k \geq 0$. Then, $\mathbf{x}^k \in \bar{B}(\mathbf{x}^0, \rho \left\|\mathbf{x}^0\right\|)$ for all k, and hence, there is a convergent sub-sequence, $\mathbf{x}^{k_j} = \mathbf{x}(t^{k_j}, \mathbf{x}^0) \to \mathbf{x}^\infty$. Thus, $\mathbf{x}^0$ positive limit point, and therefore $\Omega(\mathbf{x}^0) \neq \emptyset$.

(2): We need to show that for every $\mathbf{x}^\infty \in \Omega(\mathbf{x}^0)$, $\{\mathbf{x}(t,\mathbf{x}^0)\}_{t\geq 0} \subseteq \Omega(\mathbf{x}^0)$. Fix $\mathbf{x}^\infty \in \Omega(\mathbf{x}^0)$, then there is a sequence $\{t^k\}_{t^k \geq 0} \uparrow \infty$, such that

$$\mathbf{x}^\infty = \lim_{k\to\infty} \mathbf{x}(t^k, \mathbf{x}^0).$$

Note that by semi-group property, we know

$$\mathbf{x}(t + t^k, \mathbf{x}^0) = \mathbf{x}(t, \mathbf{x}(t^k, \mathbf{x}^0)), \forall t \geq 0$$

thus, for every $t \geq 0$,

$$\lim_{k\to\infty} \mathbf{x}(t + t^k, \mathbf{x}^0) = \lim_{k\to\infty} \mathbf{x}(t, \mathbf{x}(t^k, \mathbf{x}^0)) = \mathbf{x}(t, \mathbf{x}^\infty),$$

where we also used continuity of $\mathbf{x}(t,.)$ in its second argument. Thus, $\mathbf{x}(t,\mathbf{x}^\infty) \in \Omega(\mathbf{x}^0)$ for every $t \geq 0$.

(3): By copositivity of $-N$ on $\Gamma(\mathrm{SOL}'_{\mathrm{LCS}})$, we have

$$\phi'_{\mathbf{x}^0}(t) \leq 0, \quad \forall t \geq 0.$$

Next, we let

$$\mathbf{w}(t) := \left[\begin{array}{c} \mathbf{x}(t,\mathbf{x}^0) \\ \mathbf{u}(t,\mathbf{x}^0) \end{array}\right],$$

and note that by strict copositivity of M on $\Gamma(\mathbf{u})$,

$$\phi_{\mathbf{x}^0}(t) = \mathbf{w}(t)^T M \mathbf{w}(t) > 0.$$

Therefore, $\phi_{\mathbf{x}^0}(t)$ is monotone decreasing (non-increasing) and bounded below, and hence $\lim_{t\to\infty} \phi_{\mathbf{x}^0}(t)$ exists; let $\kappa_{\mathbf{x}^0} := \lim_{t\to\infty} \phi_{\mathbf{x}^0}(t)$ and note that for any $\mathbf{x}^\infty \in \Omega(\mathbf{x}^0)$,

$$V(\mathbf{x}^\infty, \mathbf{u}(\mathbf{x}^\infty)) = \kappa_{\mathbf{x}^0}.$$

(4): Proof of this part follows directly from parts (2) and (3). For any $\mathbf{x}^\infty \in \Omega(\mathbf{x}^0)$ we have,

$$\phi_{\mathbf{x}^\infty}(t) = V(\mathbf{x}(t, \mathbf{x}^\infty), \mathbf{u}(t, \mathbf{x}^\infty)) = \kappa_{\mathbf{x}^0}, \quad \forall t \geq 0.$$

Therefore, $\phi'_{\mathbf{x}^0} \equiv 0$. □

Now, we can go back and prove Theorem 3.7.

Proof. (Proof of Theorem 3.7): By Theorem 3.6, we have the $\mathbf{x}^e = 0$ is linearly bounded stable; that is, there exists a $\rho > 0$ such that for any $\mathbf{x}^0 \in \mathbb{R}^n$,

$$\left\|\mathbf{x}(t, \mathbf{x}^0)\right\| \leq \rho \left\|\mathbf{x}^0\right\|, \quad \forall t \geq 0.$$

Thus, $\mathbf{x}^e = 0$ is stable in the sense of Lyapunov, because given $\epsilon > 0$, we can let $\left\|\mathbf{x}^0\right\| \leq \delta := \frac{\epsilon}{\rho}$ to get, $\left\|\mathbf{x}(t, \mathbf{x}^0)\right\| \leq \epsilon$ for all $t \geq 0$. Thus, all that remains to show is the following convergence property:

for $\left\|\mathbf{x}^0\right\| \leq \delta$,

$$\mathbf{x}(t, \mathbf{x}^0) \to 0, \qquad \text{as } t \to \infty.$$

It is sufficient to show $\Omega(\mathbf{x}^0) = \{0\}$. Let $\mathbf{x}^\infty \in \Omega(\mathbf{x}^0)$; we claim that $\mathbf{x}^\infty = 0$. Note that,

$$\mathbf{z}(t, \mathbf{x}^\infty)^T N \mathbf{z}(t, \mathbf{x}^\infty) = \phi'_{\mathbf{x}^\infty}(t) = 0, \quad \forall t \geq 0,$$

where the last equality follows from part (4) of Lemma 3.8. Therefore, we get $\mathbf{x}^\infty = 0$ (by the second hypothesis of the Theorem). Thus, it follows that $\Omega(\mathbf{x}^0) = \{0\}$, which completes the proof. □

3.2 Extension of the Results to non-P Case

It is possible to extend the results of the previous section to cover the case of an LCS(A, B, C, D), where D is not a P-matrix. Such extension, which is rather non-trivial, is done in [2]. Instead of discussing the extension in detail, we only make some general comments for this case. To compensate for loss of P property of D, we make the assumption that $B\,\text{SOL}(C\mathbf{x}, D)$ is a singleton for all $\mathbf{x} \in \mathbb{R}^n$. However, this is not the end of the story. We will still assume existence of a matrix as M defined in the previous section. Moreover, we also need to add the following assumptions,

- $Q\,\text{SOL}(C\mathbf{x}, D)$ is a singleton for all $\mathbf{x} \in \mathbb{R}^n$,
- $R\,\text{SOL}(C\mathbf{x}, D)$ is a singleton for all $\mathbf{x} \in \mathbb{R}^n$.

With this added assumptions, and some further technical developments, the authors in [2] were able to extend the results in the previous section to the non-P case[5].

[5] Actually, they proved more stability results than we covered in the last section, all of which were also extended to the non-P case.

4 Concluding Remarks

We noted in our discussion in Section 3 that extending LaSalle's Theorem in the context of an LCS forces us to add several assumptions which, one may argue, are not so easy to check in general. However, as the authors in [2] commented, such practical difficulties do not stop us in developing extensions to existing theory. Although the conditions for the extensions we studied in Section 3 are difficult to check in general, it is possible to come-up with matrix theoretic results which make it easier to check these hypotheses (see for example Proposition 3.3 in [2]). Moreover, it is possible to further refine such theorems in more specialized problems with special structures.

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