A basic note on flow through porous media: single component, single phase flow

Alen Alexanderian∗

January 26, 2016

We consider the a single component single phase flow through a porous medium. Below we start from basic physical principles, and through a sequence of simplifications, arrive at the usual Poisson pressure equations. The references used in writing this are [3, 1, 2]. The starting point for our study the continuity equation, which is the statement of conservation of mass:

\[ \frac{\partial (\phi \rho)}{\partial t} + \nabla \cdot (\rho v) = q. \]

Here \( \rho \) denotes the fluid density, \( v \) is the flow velocity, \( \phi \) is the porosity, and \( q \) is a source term (inflow/outflow per unit volume at well locations). Recall that the porosity of a porous rock is the ratio between the volume of the pores and the total rock volume.

Darcy’s law (valid for low velocities) states that,

\[ v = -\frac{1}{\mu} K (\nabla p + \rho g), \]

where \( \mu \) is fluid viscosity, \( K \) is the permeability tensor (a matrix valued function), and \( g \) is the gravity vector, \( g = ge_z \) with \( e_z \) the unit vector in \( z \) direction and \( g \) the acceleration due to gravity. For simplicity, we assume an isotropic medium, \( K = \kappa I \), where \( \kappa \) is a scalar function and \( I \) is the \( 3 \times 3 \) identity matrix. Incorporating this assumption in Darcy’s law and combining Darcy’s law with the continuity equation, we get

\[ \frac{\partial (\phi \rho)}{\partial t} - \nabla \cdot \left( \rho \kappa \frac{1}{\mu} (\nabla p + \rho g) \right) = q. \]

Restricting the study of flow to a two-dimensional case, where we are concerned with horizontal flow in a plane, we may neglect the term due to gravity, and get

\[ \frac{\partial (\phi \rho)}{\partial t} - \nabla \cdot \left( \rho \kappa \frac{1}{\mu} \nabla p \right) = q. \]

(1)

The constant compressibility case In isothermal conditions, the compressibility of a fluid is defined as:

\[ c = \frac{1}{\rho^2} \frac{d \rho}{dp}. \]

∗North Carolina State University, Raleigh, NC, USA. email: alexanderian@ncsu.edu
A reasonable assumption is that of constant compressibility which applies to a variety of liquids [3]. In this case, considering

\[ \frac{d\rho}{dp} = c\rho, \]

we get that \( \rho(p) = \rho_0 \exp\{c(p - p_0)\} \), where \( \rho_0 \) is the density at a reference pressure value \( p_0 \). In this case, we also note that

\[ \rho \nabla p = \frac{1}{c} \nabla \rho, \]

and therefore, we can rewrite (1) as follows:

\[ \frac{\partial(\phi \rho)}{\partial t} - \nabla \cdot \left( \frac{\kappa}{\mu c} \nabla \rho \right) = q. \quad (3) \]

**Further simplification: incompressibility and time invariant porosity**  
Assuming a constant in time porosity and incompressibility, i.e. constant density, we can rewrite (1) as

\[ -\nabla \cdot \left( \rho \frac{\kappa}{\mu} \nabla p \right) = q, \]

which can be stated more compactly as

\[ -\nabla \cdot \left( \frac{\kappa}{\mu} \nabla p \right) = f, \]

with \( f = q/\rho \), which is the familiar form of the Poisson pressure equation.

**References**

