A basic note on flow through porous media: single component, single phase flow

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We consider the a single component single phase flow through a porous medium. Below we start from basic physical principles, and through a sequence of simplifications, arrive at the usual Poisson pressure equations. The references used in writing this are [3, 1, 2]. The starting point for our study the continuity equation, which is the statement of conservation of mass:

$$\frac{\partial(\phi\rho)}{dt} + \nabla \cdot (\rho \boldsymbol{v}) = q.$$

Here ρ denotes the fluid density, $\boldsymbol{v}$ is the flow velocity, ϕ is the porosity, and q is a source term (inflow/outflow per unit volume at well locations). Recall that the porosity of a porous rock is the ratio between the volume of the pores and the total rock volume.

Darcy's law (valid for low velocities) states that,

$$\boldsymbol{v} = -\frac{1}{\mu}\mathbf{K}(\nabla p + \rho \boldsymbol{g}),$$

where μ is fluid viscosity, $\mathbf{K}$ is the permeability tensor (a matrix valued function), and $\boldsymbol{g}$ is the gravity vector, $\boldsymbol{g} = g\boldsymbol{e}_z$ with $\boldsymbol{e}_z$ the unit vector in z direction and g the acceleration due to gravity. For simplicity, we assume an isotropic medium, $\mathbf{K} = \kappa \mathbf{I}$, where κ is a scalar function and $\mathbf{I}$ is the 3×3 identity matrix. Incorporating this assumption in Darcy's law and combining Darcy's law with the continuity equation, we get

$$\frac{\partial(\phi\rho)}{dt} - \nabla \cdot \left(\rho \frac{\kappa}{\mu}(\nabla p + \rho \boldsymbol{g})\right) = q.$$

Restricting the study of flow to a two-dimensional case, where we are concerned with horizontal flow in a plane, we may neglect the term due to gravity, and get

$$\frac{\partial(\phi\rho)}{dt} - \nabla \cdot \left(\rho \frac{\kappa}{\mu}\nabla p\right) = q. \tag{1}$$

The constant compressibility case In isothermal conditions, the compressibility of a fluid is defined as:

$$c = \frac{1}{\rho}\frac{d\rho}{dp}. \tag{2}$$

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A reasonable assumption is that of constant compressibility which applies to a variety of liquids [3]. In this case, considering

$$\frac{d\rho}{dp} = c\rho,$$

we get that $\rho(p) = \rho_0 \exp\left\{c(p - p_0)\right\}$, where ρ_0 is the density at a reference pressure value p_0. In this case, we also note that

$$\rho \nabla p = \frac{1}{c} \nabla \rho,$$

and therefore, we can rewrite (1) as follows:

$$\frac{\partial(\phi\rho)}{dt} - \nabla \cdot \left(\frac{\kappa}{\mu c} \nabla \rho\right) = q. \tag{3}$$

Further simplification: incompressibility and time invariant porosity Assuming a constant in time porosity and incompressibility, i.e. constant density, we can rewrite (1) as

$$-\nabla \cdot \left(\rho \frac{\kappa}{\mu} \nabla p\right) = q,$$

which can be stated more compactly as

$$-\nabla \cdot \left(\frac{\kappa}{\mu} \nabla p\right) = f,$$

with $f = q/\rho$, which is the familiar form of the Poisson pressure equation.

References

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