Matrix Lie groups and their Lie algebras

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Abstract

We discuss matrix Lie groups and their corresponding Lie algebras. Some common examples are provided for purpose of illustration.

1 Introduction

The goal of these brief note is to provide a quick introduction to matrix Lie groups which are a special class of abstract Lie groups. Study of matrix Lie groups is a fruitful endeavor which allows one an entry to theory of Lie groups without requiring knowledge of differential topology. After all, most interesting Lie groups turn out to be matrix groups anyway.

An abstract Lie group is defined to be a group which is also a smooth manifold, where the group operations of multiplication and inversion are also smooth. We provide a much simple definition for a matrix Lie group in Section 4. Showing that a matrix Lie group is in fact a Lie group is discussed in standard texts such as [2].

We also discuss Lie algebras [1], and the computation of the Lie algebra of a Lie group in Section 5. We will compute the Lie algebras of several well known Lie groups in that section for the purpose of illustration.

2 Notation

Let $\mathcal{V}$ be a vector space. We denote by $\mathrm{gl}(\mathcal{V})$ the space of all linear transformations on $\mathcal{V}$. If $\mathcal{V}$ is a finite-dimensional vector space we may put an arbitrary basis on $\mathcal{V}$ and identify elements of $\mathrm{gl}(\mathcal{V})$ with their matrix representation.

The following define various classes of matrices on $\mathbb{R}^n$:

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$\mathrm{gl}(n)$	:	the space of $n \times n$ real matrices (general linear algebra)
$\mathrm{sl}(n)$	:	the space of $n \times n$ real traceless matrices (special linear algebra)
$\mathrm{so}(n)$	:	the space of $n \times n$ real skew-symmetric matrices
$\mathrm{GL}(n)$	:	the group of $n \times n$ real invertible matrices (general linear group)
$\mathrm{SL}(n)$	:	the group of $n \times n$ real matrices with determinant 1 (special linear group)
$\mathrm{O}(n)$	:	the group of $n \times n$ real orthogonal matrices (orthogonal group)
$\mathrm{SO}(n)$	:	the group of $n \times n$ real orthogonal matrices with determinant 1 (special orthogonal group)

In this note we will be working with real matrix groups. Parallel developments for complex matrix groups can be done easily.

3   Topology of $\mathrm{gl}(n)$

The space $\mathrm{gl}(n)$ becomes an inner product space when equipped with the inner product,

$$\langle A, B \rangle = \mathrm{tr}(AB^T) = \sum_{i,j=1}^n A_{ij} B_{ij}, \quad \forall A, B \in \mathrm{gl}(n).$$

This inner product also induces a norm:

$$\|A\| = \langle A, A \rangle^{1/2} = \sum_{i,j=1}^n |A_{ij}|^2, \quad \forall A, B \in \mathrm{gl}(n). \tag{3.1}$$

Since $\mathrm{gl}(n)$ is finite-dimensional all norms are equivalent on $\mathrm{gl}(n)$, and hence they all induce the same topology. It is convenient to work with the norm in (3.1) as it is the analog of the Euclidean two-norm on $\mathbb{R}^n$,

$$|x| = \Big(\sum_{i=1}^n |x_i|^2 \Big)^{1/2}, \quad x \in \mathbb{R}^n.$$

Let $\{A_k\}$ be a sequence of matrices. We say, A_k converges to $A \in \mathrm{gl}(n)$ if

$$\|A_k - A\| \to 0, \text{ as } k \to \infty. \tag{3.2}$$

It is simple to see that (3.2) is equivalent to componentwise convergence:

$$(A_k)_{ij} \to (A)_{ij}, \text{ as } k \to \infty. \tag{3.3}$$

Using completeness of $\mathbb{R}$, and (3.3), we know $\mathrm{gl}(n)$ is also a complete inner product space – a Hilbert space. Let S be a subset of $\mathrm{gl}(n)$. We say S is bounded if there exists a real number $C > 0$ such that $\|A\| \leq C$ for every $A \in S$. We say S is closed if it contains all its limit points. That is, if $\{A_k\}$ is any sequence in S such that $A_k \to A$, for some $A \in \mathrm{gl}(n)$, then $A \in S$.

Finally, since $\mathrm{gl}(n)$ is finite-dimensional, by the Heine-Borel Theorem, a subset of $\mathrm{gl}(n)$ is compact if and only if it is closed and bounded.

4 Matrix Lie groups

Let us begin by the definition of a *matrix Lie group* [2]:

Definition 4.1. *A* matrix Lie group *is a subgroup $G \subseteq \mathrm{GL}(n)$ with the following property: If $\{A_k\}$ is a convergent sequence in G, $A_k \to A$ for some $A \in \mathrm{gl}(n)$, then either $A \in G$, or A is not invertible.*

Remark 4.2. *An equivalent way of definiting matrix Lie groups is to define them as closed subgroups of $\mathrm{GL}(n)$.*

4.1 Examples

Let us look at some examples of matrix Lie groups here:

(a) The general linear group $\mathrm{GL}(n)$: This is clearly a matrix Lie group.
(b) The special linear group $\mathrm{SL}(n)$: To see this is a matrix Lie group note that if $\{A_k\}$ is a sequence in $\mathrm{SL}(n)$, $\det(A_k) = 1$, such that $A_k \to A$, then by continuity of the determinant $\det(A) = 1$ also; therefore, $A \in \mathrm{SL}(n)$.
(c) The orthogonal group $\mathrm{O}(n)$: Recall that $A \in \mathrm{O}(n)$ if and only if $A^T A = I$. Now let $\{A_k\}$ be a sequence in $\mathrm{O}(n)$ such that $A_k \to A$. Passing to limit in the equation $A_k^T A_k = I$ gives that $A^T A = I$; that is, $A \in \mathrm{O}(n)$.
(d) The special orthogonal group $\mathrm{SO}(n)$: The proof that $\mathrm{SO}(n)$ is a matrix Lie group combines the arguments for $\mathrm{SL}(n)$ and $\mathrm{O}(n)$ above.

Remark 4.3. *The general linear group is not a compact group (consider for example the unbounded sequence given by $\{A_k = kI,\ k \geq 0\} \subset \mathrm{GL}(n)$). The special linear group, $\mathrm{SL}(n)$ is also non-compact (except for the trival case of $n = 1$).*

Remark 4.4. *The groups $\mathrm{O}(n)$, and $\mathrm{SO}(n)$ are compact Lie groups. Moreover, let us note that these groups are in fact closed in $\mathrm{gl}(n)$.*

5 Lie algebras

In this section, we look at Lie algebras [1] very briefly.

Definition 5.1 (Lie Algebra)**.** *A lie algebra over a field $\mathbb{F}$ is an $\mathbb{F}$-vector space $\mathcal{L}$ along with a bilinear mapping, the* Lie bracket*, $[\cdot, \cdot] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ which satisfies the following:*

(a) $[x, x] = 0, \quad \forall x \in \mathcal{L}$.
(b) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \quad \forall x, y, z \in \mathcal{L}$.

The bracket $[x, y]$ is commonly referred to as the commutator of x and y. The condition (b) in the definition is known as the Jacobi identity.

Definition 5.2 (Lie subalgebra)**.** *Given a Lie Algebra $\mathcal{L}$ a Lie subalgebra of $\mathcal{L}$ is a subspace $\mathcal{K} \subseteq \mathcal{L}$ such that*

$$[x, y] \in \mathcal{K}, \quad \forall x, y \in \mathcal{K}.$$

As an example of a Lie algebra, consider $\mathrm{gl}(\mathcal{V})$ for a finite-dimensional vector space $\mathcal{V}$, where the Lie bracket of $x, y \in \mathrm{gl}(V)$ is given by,

$$[x, y] = x \circ y - y \circ x. \tag{5.1}$$

Verifying that this satisfies properties of a Lie bracket is an straightforward exercise (though it is said to be one of those exercises every mathematician should do at least once).

In particular, $\mathrm{gl}(n)$ is a Lie algebra with the Lie bracket defined as in (5.1). As another example, $\mathrm{sl}(n)$ is also a Lie algebra when equipped with the Lie bracket in (5.1). Thus, $\mathrm{sl}(n)$ is a Lie-subalgebra of of $\mathrm{gl}(n)$.

6 The exponential map

Let $A \in \mathrm{gl}(n)$, we define the *exponential* of A, $e^A \in \mathrm{gl}(n)$ as follows,

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

The proof of convergence of the above series is standard and is well known.

The following theorem summarizes some important properties of the matrix exponential.

Theorem 6.1. *Let $A, B \in \mathrm{gl}(n)$. Then the following hold:*

   (a) $e^0 = I$.
   (b) $(e^A)^T = e^{A^T}$.
   (c) e^A is invertible and $(e^A)^{-1} = e^{-A}$.
   (d) If A and B commute, then $e^{A+B} = e^A e^B = e^B e^A$.
   (e) If $S \in \mathrm{GL}(n)$, then $e^{SAS^{-1}} = Se^A S^{-1}$.
   (f) $\det(e^A) = e^{\mathrm{tr}(A)}$.
   (g) $\frac{d}{dt} e^{tA} = A e^{tA} = e^{tA} A$.

The following theorem is also important.

Theorem 6.2. *Let $A, B \in \mathrm{gl}(n)$. Then,*

$$e^{A+B} = \lim_{k \to \infty} \left(e^{\frac{A}{k}} e^{\frac{B}{k}} \right)^k. \tag{6.1}$$

7 The Lie algebra of a Lie group

Let G be a matrix Lie group. We define, $\mathrm{Lie}(G)$ by

$$\mathrm{Lie}(G) = \{ A \in \mathrm{gl}(n) : e^{tA} \in G, \ \forall t \in \mathbb{R} \}.$$

Let us we show that $\mathrm{Lie}(G)$ is a vector subspace of $\mathrm{gl}(n)$. Let $A, B \in \mathrm{Lie}(G)$. Then, we have

$$e^{t(A+B)} = \lim_{k \to \infty} \left(e^{\frac{tA}{k}} e^{\frac{tB}{k}} \right)^k \in G,$$

where the last conclusion uses the fact that G is closed in $\mathrm{GL}(n)$. Therefore, $A + B \in \mathrm{Lie}(G)$. Also, if $A \in \mathrm{Lie}(G)$ and $\alpha \in \mathbb{R}$, then, clearly $\alpha A \in \mathrm{Lie}(G)$. Therefore, $\mathrm{Lie}(G)$ is also closed under scalar multiplication. Hence, $\mathrm{Lie}(G)$ is a vector subspace of $\mathrm{gl}(n)$.

Next, we show that $\mathrm{Lie}(G)$ is closed under the Lie bracket in (5.1). That is, for any A and B in $\mathrm{Lie}(G)$, we show $[A, B] \in \mathrm{Lie}(G)$, where $[A, B] = AB - BA$. First, we will need the following basic result.

Lemma 7.1. *Let G be a matrix Lie group. Let $A \in G$ and $X \in \mathrm{Lie}(G)$ Then, $AXA^{-1} \in \mathrm{Lie}(G)$.*

Proof. Note that, $e^{tAXA^{-1}} = Ae^{tX}A^{-1} \in G$ for every $t \geq 0$ and hence the result. $\square$

Matrix Lie groups

Now we get back at showing $[A, B] \in \text{Lie}(G)$ for all $A, B \in \text{Lie}(G)$. Let us define,

$$\Lambda(t) = e^{tA}Be^{-tA},$$

Note that by Lemma 7.1, $\Lambda(t) \in \text{Lie}(G)$ for all $t \geq 0$.

Next, we observe,
$$\Lambda'(t) = Ae^{tA}Be^{-tA} - e^{tA}BAe^{-tA}.$$

Therefore,
$$AB - BA = \Lambda'(0) = \lim_{h \to 0} \frac{\Lambda(h) - B}{h} \in \text{Lie}(G),$$

where the last conclusion follows from the already established fact that $\text{Lie}(G)$ is a vector subspace of $\text{gl}(n)$ and hence closed in norm topology of $\text{gl}(n)$. As a result of the above discussion we have the following result:

Theorem 7.2. *Let G be a matrix Lie group. Then, $\text{Lie}(G)$ is a Lie subalgebra of $\text{gl}(n)$ with the Lie bracket given in* (5.1)*.*

7.1 Examples

For the purpose of illustration, we now compute the Lie algebra of matrix Lie groups which we looked at in Section 4.

(a) The general linear group $GL(n)$: If $G = \text{GL}(n)$, then,

$$\text{Lie}(G) = \{A \in \text{gl}(n) : e^{tA} \in \text{GL}(n), \forall t \in \mathbb{R}\}$$
$$= \text{gl}(n),$$

where the last conclusion follows from the fact that e^A is invertible for all $A \in \text{gl}(n)$.

(b) The special linear group $\text{SL}(n)$: If $G = \text{SL}(n)$, then

$$\text{Lie}(G) = \{A \in \text{gl}(n) : e^{tA} \in \text{SL}(n), \forall t \in \mathbb{R}\}$$
$$= \{A \in \text{gl}(n) : \det(e^{tA}) = 1, \forall t \in \mathbb{R}\}$$
$$= \{A \in \text{gl}(n) : e^{t\,\text{tr}(A)} = 1, \forall t \in \mathbb{R}\}$$
$$= \{A \in \text{gl}(n) : \text{tr}(A) = 0\}$$
$$= \text{sl}(n).$$

(c) The orthogonal group $\text{O}(n)$: If $G = \text{O}(n)$, then

$$\text{Lie}(G) = \{A \in \text{gl}(n) : e^{tA} \in \text{O}(n), \forall t \in \mathbb{R}\}$$
$$= \{A \in \text{gl}(n) : (e^{tA^T})(e^{tA}) = I\}.$$

Now differentiating both sides of $(e^{tA^T})(e^{tA}) = I$ with respect to t gives,

$$A^T e^{tA^T}e^{tA} + e^{tA^T}Ae^{tA} = 0,$$

which at $t = 0$ reduces to,
$$A^T + A = 0.$$

Therefore, for $G = \text{O}(n)$, $\text{Lie}(G) = \text{so}(n)$.

(d) The special orthogonal group $\mathrm{SO}(n)$. If $G = \mathrm{SO}(n)$, then, using the arguments for (b) and (c) above,

$$
\begin{aligned}
\mathrm{Lie}(G) &= \{A \in \mathrm{gl}(n) : e^{tA} \in \mathrm{SO}(n),\ \forall t \in \mathbb{R}\} \\
&= \{A \in \mathrm{gl}(n) : A = -A^T, \mathrm{tr}(A) = 0\} \\
&= \{A \in \mathrm{gl}(n) : A = -A^T\} \quad (A = -A^T \Rightarrow \mathrm{tr}(A) = 0) \\
&= \mathrm{so}(n).
\end{aligned}
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References

[1] K. Erdmann and M. J. Wildon, *Introduction to Lie Algebras*, Springer, 2006.

[2] B. Hall, *Lie groups, Lie algebras, and Representations: An Elementary Introduction*, Springer, 2003.