On non-existence of Lebesgue-like measures in infinite dimension

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Abstract

It is well-known that an analogue of the Lebesgue measure cannot be defined in an infinite-dimensional Banach space. In this note, we provide a brief but self-contained proof of this fact.

1 Notation

Let $(X, \|\cdot\|)$ be a normed linear space, with $\mathbb{R}$ or $\mathbb{C}$ as the base field. In what follows $B(x, r)$ denotes the open ball centered at x with radius r. We denote by S_1 the unit sphere centered at the origin, $S_1 = \{x \in X : \|x\| = 1\}$. Moreover, for a subspace $M \subseteq X$ and a point $p \in X$, we define

$$\operatorname{dist}(p, M) = \inf_{m \in M} \|p - m\| .$$

2 Riesz's Lemma

The following result, known as Riesz's Lemma, is well-known.

Lemma 2.1 (Riesz)**.** *Let $(X, \|\cdot\|)$ be a normed linear space and suppose M is a proper closed subspace of X. Then for any $\nu \in (0, 1)$ there exists $x_\nu \in S_1$ such that $\|x_\nu - m\| \geq \nu$ for all $m \in M$.*

Proof. Let $p \in X \setminus M$ and note that $d := \operatorname{dist}(p, M) > 0$. Take $m_0 \in M$ such that $\|p - m_0\| \leq d/\nu$. Now, set $x_\nu = \frac{p - m_0}{\|p - m_0\|}$. Thus, $\|x_\nu\| = 1$ and for every $m \in M$, we have,

$$\begin{aligned} \|x_\nu - m\| &= \left\| \frac{p - m_0}{\|p - m_0\|} - m \right\| \\ &= \frac{1}{\|p - m_0\|} \left\| p - \big(m_0 + \|p - m_0\| \, m\big) \right\| \geq d/(d/\nu) = \nu. \quad \square \end{aligned}$$

The following technical result is a consequence of Riesz's Lemma. The result is stated for $B(0, 1)$ but can be easily generalized for any open ball. The following result shows why we cannot define an analogue of the Lebesgue measure in an infinite-dimensional separable Banach space.

Lemma 2.2. *Let X be an infinite dimensional normed linear space. Then there exists a countably infinite collection of disjoint balls $B(x_n, \varepsilon)$ (for some $\varepsilon > 0$) inside $B(0, 1)$.*

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Proof. Let $y_1 \in S_1$ and let $M_1 = \mathrm{span}\{y_1\}$. By Riesz's Lemma, we know there exists $y_2 \in S_1$ such that $\|y_2 - m\| \geq 1/2$ for all $m \in M_1$. We let $M_2 = \mathrm{span}\{y_1, y_2\}$ and proceeding inductively, get $y_3, y_4, y_5, \ldots$, such that $y_n \in S_1$ for all n and for subspaces

$$M_n = \mathrm{span}\{y_1, \ldots, y_n\},$$

we have $\mathrm{dist}(y_{n+1}, M_n) \geq 1/2$. Successive application of Riesz's Lemma is justified, because for all n, M_n is finite-dimensional and is thus a proper closed subspace of X. For the sequence $\{y_n\}_{n=1}^{\infty}$, we have $y_n \in S_1$ and $\|y_{n+1} - y_n\| \geq 1/2$ for all $n \in \mathbb{N}$; the latter also implies, $B(y_n, 1/4) \cap B(y_{n+1}, 1/4) = \emptyset$. Hence, the statement of the lemma holds with the collection of balls given by $\{B(x_n, \varepsilon)\}_{n=1}^{\infty}$, with $x_n = \frac{1}{2} y_n$ and $\varepsilon = 1/8$. □

3 Measures on Banach spaces

For any Borel measure μ on X to behave like the Lebesgue measure it must be a translation invariant positive measure which assigns a finite measure to open balls.

Proposition 3.1. *Let X be an infinite dimensional separable Banach space. Then there exists no non-trivial translation invariant positive Borel measure μ on X which is finite on open balls.*

Proof. Suppose μ is a translation invariant positive Borel measure which assigns finite measures to open balls. By Lemma 2.2 we know that $B(0,1)$ contains a countably infinite collection of disjoint balls $\{B(x_n, \varepsilon)\}_1^{\infty}$. Then, by translation invariance, $\mu(B(x_n, \varepsilon))$ is the same for every $n \in \mathbb{N}$, $\mu(B(x_n, \varepsilon)) = \alpha$ with $\alpha \in [0, \infty)$. If $\alpha > 0$ then we have $\mu(B(0,1)) \geq \mu(\cup_n B(x_n, \varepsilon)) = \sum_j \mu(B(x_n, \varepsilon)) = \sum_j \alpha = \infty$, which is a contradiction. Note that we also used the fact that $\{B(x_n, \varepsilon)\}_1^{\infty}$ are disjoint. On the other hand, if $\alpha = 0$, then by separability we can cover the whole space X with open balls of radius ε and get that $\mu(X) = 0$; i.e., μ is the trivial (zero) measure. □

4 Remark

The argument leading to the result on non-existence of an analogue to the Lebesgue measure in infinite-dimension is related to the argument showing that the Heine-Borel Theorem does not hold in infinite-dimensional normed linear spaces. In particular, in the argument in the proof of Lemma 2.2 we use Riesz's Lemma to construct a sequence $\{y_n\}_{n=1}^{\infty} \in \overline{B(0,1)}$ which satisfies $\|y_n - y_m\| \geq 1/2$ for $n \neq m$; clearly this sequence cannot have any convergent subsequence and thus $\overline{B(0,1)}$ is not compact. It is interesting that while the result concerning the noncompactness of the closed unit ball in infinite dimensional normed linear spaces is usually encountered, typically early on, in a first course in functional analysis, the former result, regarding the non-existence of an equivalent of a Lebesgue measure in infinite-dimensions is usually seen only in advanced treatments of probability.