On non-existence of Lebesgue-like measures in infinite dimensions

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It is well-known that an analogue of the Lebesgue measure cannot be defined in an infinite-dimensional separable normed linear spaces. Specifically, there exists no non-trivial translation invariant positive Borel measure on an infinite-dimensional separable normed linear space that assigns a finite measure to open balls. In this note, we provide a brief proof of this fact.

Introduction

The non-existence of Lebesgue-like measures in infinite-dimensional spaces is a well-known issue.[2,3] In this note, we briefly study the non-existence of non-trivial translation invariant positive Borel measures on an infinite-dimensional separable normed linear space that assign finite measures to open balls. To make the discussion broadly accessible, we recall some basics from analysis before discussing the main result under study.

[2] Y. Yamasaki. *Measures on infinite dimensional spaces*, volume 5. World Scientific, 1985

[3] B. R. Hunt, T. Sauer, and J. A. Yorke. Prevalence: a translation-invariant "almost every" on infinite-dimensional spaces. *Bulletin of the American mathematical society*, 27(2):217–238, 1992

Preliminaries

Let $(X, \|\cdot\|)$ be a normed linear space, with $\mathbb{R}$ or $\mathbb{C}$ as the base field. In what follows $B(x, r)$ denotes the open ball centered at x with radius r. Also, we denote by S_1 the unit sphere centered at the origin.[4] Moreover, for a subspace $M \subseteq X$ and a point $p \in X$, we define

[4] $S_1 = \{x \in X : \|x\| = 1\}$.

$$\mathrm{dist}(p, M) = \inf_{m \in M} \|p - m\|.$$

The following result, known as Riesz's Lemma, is well-known.[5]

[5] E. Kreyszig. *Introductory functional analysis with applications*, volume 81. wiley New York, 1989

Lemma 1. (Riesz) Let $(X, \|\cdot\|)$ be a normed linear space and suppose M is a proper closed subspace of X. Then for any $\nu \in (0, 1)$ there exists $x_\nu \in S_1$ such that $\|x_\nu - m\| \geq \nu$ for all $m \in M$.

Proof. Let $p \in X \setminus M$ and note that $d := \mathrm{dist}(p, M) > 0$. Take $m_0 \in M$ such that $\|p - m_0\| \leq d/\nu$. Now, set $x_\nu = \frac{p - m_0}{\|p - m_0\|}$. Thus, $\|x_\nu\| = 1$ and for every $m \in M$, we have,

$$\begin{aligned}\|x_\nu - m\| &= \left\| \frac{p - m_0}{\|p - m_0\|} - m \right\| \\ &= \frac{1}{\|p - m_0\|} \left\| p - \big(m_0 + \|p - m_0\| \, m\big) \right\| \geq d/(d/\nu) = \nu. \quad \square\end{aligned}$$

The main result

The following technical result is a consequence of Riesz's Lemma. The result is stated for $B(0,1)$ but can be easily generalized for any open ball. As seen shortly, this result shows why we cannot define an analogue of the Lebesgue measure in an infinite-dimensional separable normed linear space.[6]

[6] A normed linear space is separable if it contains a countable dense subset.

LEMMA 2. Let X be an infinite dimensional normed linear space. For every $\varepsilon > 0$, then there exists a countably infinite collection of disjoint balls $B(x_n, \varepsilon)$.

Proof. Let $y_1 \in S_1$ and let $M_1 = \text{span}\{y_1\}$. By Riesz's Lemma, we know there exists $y_2 \in S_1$ such that $\|y_2 - m\| \geq 1/2$ for all $m \in M_1$. We let $M_2 = \text{span}\{y_1, y_2\}$ and proceeding inductively, get $y_3, y_4, y_5, \ldots$, such that $y_n \in S_1$ for all n and for subspaces

$$M_n = \text{span}\{y_1, \ldots, y_n\},$$

we have $\text{dist}(y_{n+1}, M_n) \geq 1/2$. Successive application of Riesz's Lemma is justified, because for all n, M_n is finite-dimensional and is thus a proper closed subspace of X. For the sequence $\{y_n\}_{n=1}^{\infty}$, we have $y_n \in S_1$ and $\|y_{n+1} - y_n\| \geq 1/2$ for all $n \in \mathbb{N}$; the latter also implies, $B(y_n, 1/4) \cap B(y_{n+1}, 1/4) = \emptyset$. Hence, the statement of the lemma holds with the collection of balls given by $\{B(x_n, \varepsilon)\}_{n=1}^{\infty}$, with $x_n = \frac{1}{2}y_n$ and $\varepsilon = 1/8$. □

We refer to measures on $(X, \mathcal{B}(X))$, where $\mathcal{B}(X)$ is the Borel sigma-algebra on X, as Borel measures on X. For a Borel measure μ on X to behave like the Lebesgue measure it must be a translation invariant positive measure that assigns a finite measure to open balls. The following result shows we cannot define such a measure on an infinite-dimensional separable normed linear space.

THEOREM 1. Let X be an infinite dimensional separable normed linear space. Then there exists no non-trivial translation invariant positive Borel measure μ on X that is finite on open balls.

Proof. Suppose μ is a translation invariant positive Borel measure that assigns a finite measure to each open ball in X. By Lemma 2 we know that $B(0,1)$ contains a countably infinite collection of disjoint balls $\{B(x_n, \varepsilon)\}_1^{\infty}$. Then, by translation invariance, $\mu(B(x_n, \varepsilon))$ is the same for every $n \in \mathbb{N}$. That is, $\mu(B(x_n, \varepsilon)) = \alpha$ with $\alpha \in [0, \infty)$ a constant. If $\alpha > 0$, then $\mu(B(0,1)) \geq \mu(\cup_n B(x_n, \varepsilon)) = \sum_n \mu(B(x_n, \varepsilon)) = \sum_n \alpha = \infty$, which is a contradiction. Note that we also used the fact that $\{B(x_n, \varepsilon)\}_1^{\infty}$ are disjoint. On the other hand, if $\alpha = 0$, then by separability we can cover the whole space X with countably infinite family of open balls of radius ε and get that $\mu(X) = 0$; i.e., μ is the trivial (zero) measure. □

Remark

The argument leading to the result on non-existence of an analogue to the Lebesgue measure in infinite dimensions is related to the one showing the Heine–Borel Theorem does not hold in infinite-dimensional normed linear spaces. In particular, in the argument in the proof of Lemma 2, we use Riesz's Lemma to construct a sequence $\{y_n\}_{n=1}^{\infty} \in \overline{B(0,1)}$, which satisfies $\|y_n - y_m\| \geq 1/2$ for $n \neq m$. Clearly, this sequence cannot have any convergent subsequence and thus $\overline{B(0,1)}$ is not compact. It is interesting that while the result concerning the non-compactness of the closed unit ball in infinite dimensional normed linear spaces is usually seen early on in a first course in functional analysis, the former result, regarding the non-existence of an analogue of a Lebesgue measure in infinite dimensions is usually seen only in advanced treatments of probability.

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References

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