Optimization in infinite-dimensional Hilbert spaces

Alen Alexanderian[*]

Abstract

We consider the minimization of convex functionals on a real Hilbert space. The requisite theory is reviewed, and concise proofs of the relevant results are given.

1 Introduction

We are concerned here with the classical results on optimization of convex functionals in infinite-dimensional real Hilbert spaces. When working with infinite-dimensional spaces, a basic difficulty is that, unlike the case in finite-dimension, being closed and bounded does not imply that a set is compact. In reflexive Banach spaces, this problem is mitigated by working in weak topologies and using the result that the closed unit ball is weakly compact. This in turn enables mimicking some of the same ideas in finite-dimensional spaces when working on unconstrained optimization problems.

It is the goal of these note to provide a concise coverage of the problem of minimization of a convex function on a Hilbert space. The focus is on real Hilbert spaces, where there is further structure that makes some of the arguments simpler. Namely, proving that a closed and convex set is also weakly sequentially closed can be done with an elementary argument, whereas to get the same result in a general Banach space we need to invoke Mazur's Theorem. The ideas discussed in this brief note are of great utility in theory of PDEs, where weak solutions of problems are sought in appropriate Sobolev spaces.

After a brief review of the requisite preliminaries in Sections 2– 4, we develop the main results we are concerned with in Section 5. Though, the results in this note are classical, we provide proofs of key theorems for a self contained presentation. A simple application, regarding the Dirichlet problem, is provided in Section 6 for the purposes of illustration.

2 Weak convergence

In general, for a Banach space X, we know by Banach-Alaoglu Theorem that the closed unit ball in X^* is compact in weak* topology. Furthermore, if X is a reflexive Banach space, then we get as a consequence of Banach-Alaoglu Theorem that closed unit ball in X is weakly compact. Hence, in particular, in a Hilbert space, we know that any bounded sequence has a weakly convergent subsequence.

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[*]North Carolina State University, Raleigh, NC, USA. E-mail: `alexanderian@ncsu.edu`
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Let us recall that a sequence $\{x_n\}$ in a Hilbert space H converges weakly to x if,

$$\lim_{n\to\infty} \langle x_n, u\rangle = \langle x, u\rangle, \quad \forall u \in H.$$

We use the notation $x_n \rightharpoonup x$ to mean that x_n converges weakly to x.

Before moving further we recall an important point about notions of compactness and sequential compactness in weak topologies. It is common knowledge that compactness and sequential compactness are equivalent in metric spaces. The situation is not obvious in the case of weak topology of an infinite-dimensional normed linear space. In fact, in infinite-dimensional normed linear spaces, the weak topology is not metrizable. However, there is an analogous result to metric spaces relating the ideas of weak compactness and sequential weak compactness in weak topologies. The well known Eberlein-Šmulian Theorem (see e.g. [1, 2]), states that in Banach spaces the notions of weak compactness and sequential weak compactness are equivalent.

3   Lower Semi-continuous Functions

We start by looking at the definition of lower semi-continuous functions.

Definition 3.1. *A function real valued function f on a Banach space is lower semi-continuous (lsc) if*

$$f(x) \le \liminf_{n\to\infty} f(x_n)$$

for all sequences $\{x_n\}$ in X such that $x_n \to x$ (strongly).

Similarly, we define a weakly (sequentially) lower semi-continuous function as below.

Definition 3.2. *A function f is weakly sequentially lower semi-continuous (weakly lsc) if*

$$f(x) \le \liminf_{n\to\infty} f(x_n)$$

for all sequences $\{x_n\}$ such that $x_n \rightharpoonup x$.

The following result, which we state without proof is useful when working with lsc (or weakly sequentially lsc) functions.

Theorem 3.3. *Let X be a Banach space and $f : X \to \mathbb{R}$. Then the following are equivalent.*

(a) f is (weakly sequentially) lsc.

(b) $\mathrm{epi}(f)$, is (weakly sequentially) closed.

Here $\mathrm{epi}(f)$ denotes the epigraph of the function f:

$$\mathrm{epi}(f) := \{(x, r) \in \mathrm{dom}(f) \times \mathbb{R} : f(x) \le r\}.$$

4   Convex Functions

We recall the definition of a convex function.

Definition 4.1. *Let X be a metric space and $C \subseteq X$ a non-empty convex set. A function $f : C \to \mathbb{R}$ is convex if $\forall \alpha \in [0, 1]$ and $\forall x, y \in C$*

$$f(\alpha x + (1 - \alpha)y) \le \alpha f(x) + (1 - \alpha)f(y).$$

Note that the function f in the above definition is called strictly convex if the above inequality is strict for $x \neq y$ and $\alpha \in (0,1)$. We end this section by recalling the following useful characterization of convex functions.

Lemma 4.2. *Let X be a metric space and $C \subseteq X$ a non-empty convex set. A function $f : C \to \mathbb{R}$ is convex if and only if $\mathrm{epi}(f)$ is convex.*

Let us also recall the notion of a coercive function, as we will be considering such functions shortly.

Definition 4.3. *Let X be a Banach space and let f be a real valued function on X; that is, $f : X \to \mathbb{R}$. The function f is called coercive if the following holds:*

$$\lim_{\|x\| \to \infty} f(x) = \infty.$$

5 Optimization in a Hilbert space

In this section, we derive a generalized Weierstrass Theorem which gives the criteria for existence of a minimizer for a function on a Hilbert space. The discussion is focused on real Hilbert spaces. In what follows $\mathcal{H}$ will denote a real Hilbert space.

5.1 Closed convex sets in a Hilbert space

The following result is of great utility in what follows.

Lemma 5.1. *Let $K \subseteq \mathcal{H}$ be a (strongly) closed and convex set. Then, K is weakly sequentially closed.*

Proof. Let $\{x_n\}$ be a sequence in K and suppose $x_n \rightharpoonup x^*$. We show $x* \in K$ by showing $x^* = \Pi_K(x^*)$, where $\Pi_K(x^*)$ denotes the projection of x^* into the closed convex set K. Recall that the projection $\Pi_K(x^*)$ satisfies the variational inequality, $\langle x^* - \Pi_K(x^*), y - \Pi_K(x^*) \rangle \leq 0$ for all $y \in K$. Therefore,

$$\langle x^* - \Pi_K(x^*), x_n - \Pi_K(x^*) \rangle \leq 0, \quad \forall n. \tag{5.1}$$

Next, note that since $x_n \rightharpoonup x^*$, we have,

$$\|x^* - \Pi_K(x^*)\|^2 = \langle x^* - \Pi_K(x^*), x^* - \Pi_K(x^*) \rangle$$
$$= \lim_{n \to \infty} \langle x^* - \Pi_K(x^*), x_n - \Pi_K(x^*) \rangle$$

Therefore, in view of (5.1), we have that $\|x^* - \Pi_K(x^*)\| = 0$, that is, $x^* = \Pi_K(x^*)$ and the proof is complete. $\square$

An immediate consequence of Lemma 5.1 is given by:

Corollary 5.2. *Let $f : \mathcal{H} \to \mathbb{R}$ a lsc convex function. Then, f is weakly lsc also.*

Proof. Since f is convex, $\mathrm{epi}(f)$ is convex. Moreover, because f is (strongly) lsc, $\mathrm{epi}(f)$ is strongly closed. Hence, we can apply the previous lemma to get that $\mathrm{epi}(f)$ is weakly sequentially closed also, which in turn implies that f is weakly sequentially lsc. $\square$

5.2 Optimization in a Hilbert Space

The following generalized Weierstrass Theorem is in a sense the main point of this note. The proof given here is classical, and follows in similar lines as that given in [3].

Theorem 5.3. *Suppose $C \subseteq \mathcal{H}$ is a weakly sequentially closed and bounded set. Suppose $f : C \to \mathbb{R}$ is weakly sequentially lsc. Then f is bounded from below and has a minimizer on C.*

Proof. First we show f is bounded from below. Suppose to the contrary that f is not bounded from below. Then there exist a sequence $\{x_n\} \in C$ such that $f(x_n) < -n$ for all n. Now since C is bounded $\{x_n\}$ has a weakly convergent subsequence $\{x_{n_k}\}$, $x_{n_k} \rightharpoonup x^*$. Moreover, C is weakly sequentially closed and hence $x^* \in C$. Then, since f is weakly sequentially lsc we have $f(x^*) \leq \liminf f(x_{n_k}) = -\infty$ which is a contradiction. Hence, f is bounded from below.

Next, we show the existence of a minimizer. Let $\{x_n\} \in C$ be a minimizing sequence for f; that is $f(x_n) \to \inf_C f(x)$. Let $\alpha := \inf_C f(x)$. Since C is bounded and weakly sequentially closed, it follows that $\{x_n\}$ has a weakly convergent subsequence $x_{n_k} \rightharpoonup x^* \in C$. Next, since f is weakly sequentially lsc we have

$$\alpha \leq f(x^*) \leq \liminf f(x_{n_k}) = \lim f(x_{n_k}) = \alpha.$$

Hence, $f(x^*) = \alpha$ and the Theorem is proved. $\qquad\square$

Remark 5.4. *The assertions of the above theorem remain valid in reflexive Banach spaces.*

The following Theorem involves a strongly lsc convex function on a strongly closed and convex set.

Theorem 5.5. *Let C be a convex, strongly closed, and bounded subset of $\mathcal{H}$. Suppose, $f : C \to \mathbb{R}$ is a strongly lsc and convex function. Then f is bounded from below and attains a minimizer on C.*

Proof. The idea of the proof is to show that the hypotheses of Theorem 5.3 hold. C is strongly closed and convex and hence by Lemma 5.1 is also weakly sequentially closed. Moreover, Since f is strongly lsc and convex, it is also weakly lsc by Corollary 5.2. Thus, we have $f : C \to \mathbb{R}$ weakly lsc and C a weakly closed and bounded set in H which allows us to apply the Generalized Weierstrass Theorem to conclude that f is bounded from below and attains a minimizer on C. $\qquad\square$

Note that in the above Theorem, if f is strictly convex the minimizer will be unique; this follows by noting that if we had two distinct minimizers u_1 and u_2 in C, $f(u_1) = f(u_2) = \inf_{u \in C} f(u)$, then, by strict convexity of f, we would get $f((u_1 + u_2)/2) < f(u_1)$, which is a contradiction.

Finally, we have the following result on strongly lsc convex coercive functions on a Hilbert space.

Corollary 5.6. *Let $f : \mathcal{H} \to \mathbb{R}$ be a strongly lsc, convex, and coercive function. Then f is bounded from below and attains a minimizer.*

Proof. Under the assumptions of the corollary, it is straightforward to note that f is bounded from below. Next, fix a $\delta > 0$; since f is coercive, there exists $M \in \mathbb{R}$ such that $f(x) \geq \inf_H f(y) + \delta$ for all $x \in \{x \mid \|x\| > M\}$. Then consider $f : C \to \mathbb{R}$ with $C = \{x \mid \|x\| \leq M\}$ and apply the previous Theorem. $\qquad\square$

6 A simple application

Consider the Dirichlet problem,

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and $f \in L^2(\Omega)$. It is well known that the weak solution of this problem is the solution to

$$\min_{u \in H_0^1(\Omega)} J(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega f u \, dx$$

It is straightforward to note that J is convex and continuous, and coercive (coercivity of J is a consequence of Poincare-Friedrich inequality). Thus, the existence of a unique minimizer is ensured by application of Corollary 5.6.

References

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