Groups of linear operators on Banach spaces

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Abstract

Unitary groups of linear operators arise naturally in connection with measure preserving flows on probability spaces. The purpose of this note is to provide a brief but precise exposure to the subject and collect some useful references. In this note, the focuse will be on one-parameter families of operators.

1 Introduction

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2 Basic definitions

In what follows $(X, \|\cdot\|)$ is a Banach space.

Definition 2.1 (Strongly continuous semigroup)**.** *Suppose a family of bounded linear operators $\{\Gamma_t\}_{t\in\mathbf{R}_+}$, $\Gamma_t : X \to X$, satisfies the following:*

1. *$\Gamma_0 = I$, where I denotes the identity map on X.*
2. *$\Gamma_{s+t} = \Gamma_s \circ \Gamma_t = \Gamma_t \circ \Gamma_s$, for all $s, t \in \mathbf{R}_+$.*
3. *The mapping $t \mapsto \Gamma_t u$ from $\mathbf{R}_+$ to X is continuous for all $u \in X$.*

Then, we call $\{\Gamma_t\}_{t\in\mathbf{R}_+}$ a strongly continuous semigroup of linear operators.

The third item in the above definition is the continuity property. Note that using the semigroup property (the second item in the definition), it is sufficient to verify continuity at $t = 0$, that is we say a group of operators is strongly continuous if,

$$\lim_{t\to 0} \|\Gamma_t u - u\| = 0, \quad \text{for all } u \in X.$$

If we replace $\mathbf{R}_+$ by $\mathbf{R}$ in Definition 2.1 we will get a strongly continuous group of linear operators.

Definition 2.2 (Infinitesimal generator)**.** *Given a group $\{\Gamma_t\}_{t\in\mathbf{R}}$ of bounded linear operators on X define the operator A as follows:*

$$Au = \lim_{t\to 0} \frac{\Gamma_t u - u}{t},$$

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for $u \in \mathcal{D}(A)$, where $\mathcal{D}(A) \subseteq X$ is the set of all u such that the above limit exists. We call the operator $A : \mathcal{D}(A) \to X$ the infinitesimal generator of the group $\{\Gamma_t\}_{t\in\mathbf{R}}$.

3 Strongly continuous groups

The following is the most basic result regarding strongly continuous groups of operators [3].

Theorem 3.1. *Let X be a Banach space, and let $\{\Gamma_t\}_{t\in\mathbf{R}}$ be a strongly continuous group of linear operators on X, with the infinitesimal generator A. Then, for every $u \in \mathcal{D}(A)$ we have,*

1. *$\Gamma_t u \in \mathcal{D}(A)$, for all $t \in \mathbf{R}$,*
2. *$A\Gamma_t u = \Gamma_t A u$, for all $t \in \mathbf{R}$,*
3. *The mapping $t \mapsto \Gamma_t u$ is differentiable with*

$$\frac{d}{dt}\Gamma_t u = A\Gamma_t u, \quad \forall\, t \in \mathbf{R}.$$

Proof. Let u be in $\mathcal{D}(A)$, and fix $t \in \mathbf{R}$. Note that,

$$\begin{aligned} A\Gamma_t u &= \lim_{s\to 0} \frac{\Gamma_s\Gamma_t u - \Gamma_t u}{s} \\ &= \lim_{s\to 0} \frac{\Gamma_t\Gamma_s u - \Gamma_t u}{s} \\ &= \lim_{s\to 0} \frac{\Gamma_t(\Gamma_s u - u)}{s} \\ &= \Gamma_t \left[\lim_{s\to 0} \frac{\Gamma_s u - u}{s}\right] \\ &= \Gamma_t A u, \end{aligned}$$

which establishes the first and second statements of the theorem. Next, for a fixed (but arbitrary) $u \in \mathcal{D}(A)$, define $\phi_u(t) = \Gamma_t u$. we will show that for all $t \in \mathbf{R}$, the derivative $\phi'(t)$ exists and $\phi'(t) = A\Gamma_t u$. Let $t \in \mathbf{R}$ be fixed but arbitrary.

$$\begin{aligned} \lim_{h\to 0} \frac{\phi_u(t+h) - \phi_u(t)}{h} &= \lim_{h\to 0} \frac{\Gamma_{t+h} u - \Gamma_t u}{h} \\ &= \lim_{h\to 0} \frac{\Gamma_t\Gamma_g u - \Gamma_t u}{t} \\ &= \Gamma_t \left[\lim_{h\to 0} \frac{\Gamma_h u - u}{h}\right] \\ &= \Gamma_t A u, \end{aligned}$$

Thus, we have

$$\phi'(t) = \frac{d}{dt}\Gamma_t u = \Gamma_t A u = A\Gamma_t u,$$

where the last equality follows from the second statement of the theorem proved earlier. $\square$

4 Unitary groups of operators on Hilbert spaces

Let us now work in a Hilbert space setting. Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space. Let $\{\Gamma_t\}_{t\in\mathbf{R}}$ be a group of linear operators on H; we say the group is unitary if,

$$\langle \Gamma_t u, \Gamma_t v\rangle = \langle u, v\rangle, \quad \forall\, u, v \in H.$$

The following simple result is worth noting.

Lemma 4.1. *The infinitesimal generator of a unitary group of operators is skew symmetric. That is, if $\{\Gamma_t\}_{t\in\mathbf{R}}$ is a unitary group on a Hilbert space H, and $A : \mathcal{D}(A) \to H$ is its infinitesimal generator we have,*

$$\langle Au, v\rangle = -\langle u, Av\rangle.$$

Proof. Let u, v be in $\mathcal{D}(A)$, we have

$$\begin{aligned}
\langle Au, v\rangle &= \langle \lim_{h\to 0} \frac{\Gamma_h u - u}{h}, v\rangle \\
&= \lim_{h\to 0} \frac{1}{h}\langle \Gamma_h u - u, v\rangle \\
&= \lim_{h\to 0} \frac{1}{h}\big[\langle \Gamma_h u, v\rangle - \langle u, v\rangle\big] \\
&= \lim_{h\to 0} \frac{1}{h}\big[\langle u, \Gamma_{-h} v\rangle - \langle u, v\rangle\big] \\
&= \lim_{h\to 0} \frac{1}{h}\langle u, \Gamma_{-h} v - v)\rangle \\
&= -\langle u, Av\rangle.
\end{aligned}$$

□

5 Measure preserving flows

Definition 5.1. *Let $(\Omega, \boldsymbol{\mathcal{F}}, \mu)$ be a measure space. A bijection $\phi : \Omega \to \Omega$ is called an automorphism if for every $F \in \boldsymbol{\mathcal{F}}$ we have $\phi(F) \in \boldsymbol{\mathcal{F}}$, $\phi^{-1}(F) \in \boldsymbol{\mathcal{F}}$, and*

$$\mu(F) = \mu\big(\phi(F)\big) = \mu\big(\phi^{-1}(F)\big).$$

Let us recall the definition of a flow [2, 5] on a measure space.

Definition 5.2. *Let $(\Omega, \boldsymbol{\mathcal{F}}, \mu)$ be a measure space. Let $\{T_x\}_{x\in\mathbf{R}}$ be a one-parameter group of automorphisms on $(\Omega, \boldsymbol{\mathcal{F}}, \mu)$. We say $\{T_x\}_{x\in\mathbf{R}}$ is a flow if for every measurable function f on Ω, the composition $f \circ T_x : \Omega \times \mathbf{R} \to \mathbf{R}$ is measurable in $(\Omega, \boldsymbol{\mathcal{F}}) \otimes (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ where $\mathcal{B}(\mathbf{R})$ denotes the Borel σ-algebra on $\mathbf{R}$.*

Here we look at an example of unitary group of operators which arises naturally when working with flows on a probability space. Let $(\Omega, \boldsymbol{\mathcal{F}}, \mu)$ be a probability space and let $\{T_x\}_{x\in\mathbf{R}}$ be a one-dimensional flow on Ω. Define,

$$U_x f = f \circ T_x, \quad x \in \mathbf{R}. \tag{5.1}$$

Then, $\{U_x\}_{x\in\mathbf{R}}$ is a unitary group of operators. The group property is simple to show. To show unitarity we first recall that due to measure preserving property of $\{T_x\}$

$$\int_\Omega f(\omega)\, d\mu(\omega) = \int_\Omega f\big(T_x(\omega)\big)\, d\mu(\omega).$$

The proof of this result is simple; see for example [6]. Then, we can show unitarity of $\{U_x\}_{x\in\mathbf{R}}$ as follows:

$$\begin{aligned}
\langle U_x u, U_x v\rangle &= \int_\Omega (U_x u)(\omega)(U_x v)(\omega)\, d\mu(\omega) \\
&= \int_\Omega u\big(T_x(\omega)\big) v\big(T_x(\omega)\big)\, d\mu(\omega) \\
&= \int_\Omega u(\omega) v(\omega)\, d\mu(\omega) \\
&= \langle u, v\rangle.
\end{aligned}$$

Finally we have the following result.

Theorem 5.3. *The group of operators* $\{U_x\}$ *defined above is strongly continuous.*

A proof of the above Theorem can be found in [2, 4]. The proof in [2] uses the following result of Von Neumann [1932] (see [1] for a proof):

Theorem 5.4. *Let* H *be a separable Hilbert space, and let* $\{U_x\}_{x\in\mathbf{R}}$ *be a group of unitary operators on* H. *If the mapping* $t \mapsto \langle U_t f, g\rangle$ *is Lebesgue measurable for every* $f, g \in H$, *the* $\{U_t\}$ *is a strongly continuous group of operators.*

The proof in [4] is a simpler alternative which uses Fubini's Theorem along with the Dominated Convergence Theorem.

References

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