Gershgorin Circles

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Abstract

Gershgorin's theorem is a well-known result that enables localizing the eigenvalues of a matrix. We present a brief proof of the theorem and, for illustration, consider a couple simple applications of the result.

Gershgorin circles provide a basic means of localizing eigenvalues of a matrix. This is made precise by the Gershgorin Theorem. Below, we provide a concise proof of the result. The basic argument is standard; see, e.g., [1, 2]. We also consider a couple of simple application of the theorem.

1 Gershgorin's theorem

Theorem 1.1. *Let* $\mathbf{A} = (a_{ij}) \in \mathbb{C}^{n\times n}$. *Define the disks* R_i *as follows,*

$$R_i := \{z \in \mathbb{C} : |a_{ii} - z| \le \sum_{j\neq i} |a_{ij}|\}.$$

Then,

(a) every eigenvalue of $\mathbf{A}$ *lies in the union* $S = \cup_{i=1}^n R_i$; *and*

(b) if $\widehat{S}$ *is a union of* m *disks* R_i, *such that* $\widehat{S}$ *is disjoint from all other disks, then* $\widehat{S}$ *contains precisely* m *eigenvalues of* $\mathbf{A}$ *(counting multiplicities).*

Proof. We begin by proving (a). Consider an eigenpair $(\lambda, \boldsymbol{v})$ of $\mathbf{A}$. Let i be chosen according to:

$$|v_i| = \max_{1\le j\le n} |v_j|.$$

Since $\boldsymbol{v}$ is an eigenvector, it is nonzero. Therefore, $|v_i| \neq 0$. Also, since $\mathbf{A}\boldsymbol{v} = \lambda\boldsymbol{v}$, we have $\sum_{j=1}^n a_{ij}v_j = \lambda v_i$. Therefore,

$$(\lambda - a_{ii})v_i = \sum_{j\neq i} a_{ij}v_j,$$

and thus

$$|\lambda - a_{ii}||v_i| \le \sum_{j\neq i} |a_{ij}||v_j| \le \big(\sum_{j\neq i} |a_{ij}|\big)|v_i|.$$

Dividing through by $|v_i|$ gives $|\lambda - a_{ii}| \le \sum_{j\neq i} |a_{ij}|$. That is, $\lambda_i \in R_i$, and hence, the first part of the theorem is proved.

For the second part of the theorem we proceed as follows. Define,

$$\mathbf{D} := \mathrm{diag}(a_{11}, \ldots, a_{nn}) \quad \text{and} \quad \mathbf{E} := \mathbf{A} - \mathbf{D}.$$

For $\varepsilon \in [0, 1]$ define,

$$\mathbf{M}(\varepsilon) := \mathbf{D} + \varepsilon\mathbf{E}.$$

Note that $\mathbf{M}(0) = \mathbf{D}$ and $\mathbf{M}(1) = \mathbf{A}$. Also, define $R_i(\varepsilon)$ as follows,

$$R_i(\varepsilon) := \{z \in \mathbb{C} : |a_{ii} - z| \le \sum_{j\neq i} \varepsilon|a_{ij}|\}.$$

Without loss of generality, we may assume $\widehat{S} = \cup_{i=1}^{m} R_i$. Let $\widehat{S}(\varepsilon) = \cup_{i=1}^{m} R_i(\varepsilon)$, and $\widetilde{S}(\varepsilon) = \cup_{i=m+1}^{n} R_i(\varepsilon)$. By assumption $\widehat{S}(1) \cap \widetilde{S}(1) = \emptyset$. Thus, since the circles $R_i(\varepsilon)$ shrink as $\varepsilon \to 0$, it follows that $\widehat{S}(\varepsilon) \cap \widetilde{S}(\varepsilon) = \emptyset$ for every $\varepsilon \in [0,1]$. In particular, $\widehat{S}(0)$ contains exactly m eigenvalues, $a_{11}, \ldots, a_{mm}$, of $\mathbf{M}(0) = \mathbf{D}$. Next, note that by the first statement of the theorem, all eigenvalues $\lambda_1(\varepsilon), \ldots, \lambda_n(\varepsilon)$ of $\mathbf{M}(\varepsilon)$ are contained in $\widehat{S}(\varepsilon) \cup \widetilde{S}(\varepsilon)$. Thus, since $\widehat{S}(\varepsilon)$ and $\widetilde{S}(\varepsilon)$ are disjoint for all $\varepsilon \in [0,1]$, and since eigenvalues $\lambda_i(\varepsilon)$ depend continuously on ε, it follows that as ε increases from 0 to 1, $\lambda_i(\varepsilon)$ remain in their respective partition. [1] Therefore, since $\widehat{S}(0)$ contains exactly m eigenvalues of $\mathbf{M}(0)$, it follows that $\widehat{S}(1) = \widehat{S}$ contains exactly m eigenvalues of $\mathbf{M}(1) = \mathbf{A}$. □

[1] That is, the eigenvalues cannot jump from $\widehat{S}(\varepsilon)$ to $\widetilde{S}(\varepsilon)$.

2 A couple of simple applications

We first consider the following simple corollary of Gershgorin's theorem.

Corollary 2.1. *Suppose that $A \in \mathbb{R}^{n\times n}$ has a Gershgorin circle R that is disjoint from the other circles. Then, the eigenvalue contained in R must be real.*

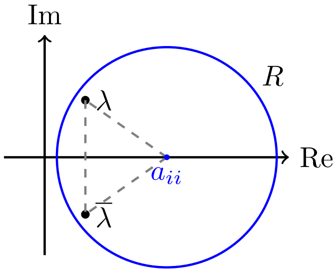
Proof. This can be shown as follows. Suppose R is centered at the ith diagonal entry $a_{ii} \in \mathbb{R}$ of $\mathbf{A}$, for some $i \in \{1, \ldots, n\}$. Suppose to the contrary that R contains a complex eigenvalue $\lambda = x + iy$. Note that in this case $\bar{\lambda}$ is also an eigenvalue.[2] Furthermore,

$$|a_{ii} - \bar{\lambda}|^2 = (a_{ii} - x)^2 + (-y)^2 = (a_{ii} - x)^2 + y^2 = |a_{ii} - \lambda|^2.$$

Therefore, $\bar{\lambda} \in R$ as well.[3] This, however, contradicts the fact that R contains only one eigenvalue of $\mathbf{A}$. □

[2] This follows from the fact that the characteristic polynomial, $p(z) = \det(z\mathbf{I} - \mathbf{A})$ of $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a polynomial with real coefficients. Hence, $p(\bar{\lambda}) = \overline{p(\lambda)} = 0$. Thus, $\bar{\lambda}$ is an eigenvalue of $\mathbf{A}$.

[3] The example image below illustrates what happens if R contains a complex eigenvalue λ.



The following is also a simple consequence of Gershgorin's theorem.

Corollary 2.2. *Strictly diagonally dominant matrices are nonsingular.*

Proof. Assume that $\mathbf{A}$ is a strictly diagonally dominant matrix. That is,

$$|a_{ii}| > \sum_{j\neq i} |a_{ij}|, \quad i \in \{1, \ldots, n\}.$$

This implies that zero does not belong to any of the Gershgorin circles of $\mathbf{A}$. Therefore, zero is not an eigenvalue of $\mathbf{A}$ and hence $\mathbf{A}$ is nonsingular. □

References

[1] Kendall E. Atkinson. *An introduction to numerical analysis*. John Wiley & Sons Inc., New York, second edition, 1989.

[2] James M. Ortega. *Numerical analysis. A second course*. Academic Press, New York, 1972. Computer Science and Applied Mathematics.