A brief note on Gaussian quadrature

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Abstract

We review some key results regarding Gaussian quadrature.

1 A key result

Let $I_n(f)$ be an $n$-point quadrature formula

$$I_n(f) = \sum_{j=1}^{n} w_j f(x_j),$$

where $x_j$ are quadrature nodes, and $w_j$ are quadrature weights. The rule $I_n(f)$ is used to approximate $I(f) = \int_{a}^{b} f(x)w(x) \, dx$, where $w \geq 0$ is a weight function. We denote by $E_n(f) = I(f) - I_n(f)$ the quadrature error. The following result from [1] reveals some connections between quadrature and ideas from approximation theory. The argument presented here follows in similar lines as that in [1].

**Theorem 1.1.** Let $I_n(f) = \sum_{j=1}^{n} w_j f(x_j)$ be an $n$-point quadrature formula for approximating $I(f) = \int_{a}^{b} f(x)w(x) \, dx$, where $w \geq 0$ is a weight function. Define,

$$\gamma_n(x) = \prod_{k=1}^{n} (x - x_k).$$

Given $0 \leq k \leq n$, we have

$$E_n(p) = 0, \quad \forall p \in \mathbb{P}_{n-1+k}, \quad (1.1)$$

if and only if the following hold:

(a) $I_n$ is interpolatory.

(b) $\int_{a}^{b} \gamma_n(x)p(x)w(x) \, dx = 0, \quad \forall p \in \mathbb{P}_{k-1}.$

**Proof.** Suppose (1.1) holds. This immediately implies (a). To see (b), notice that for $p \in \mathbb{P}_{k-1}$, $\gamma_n p \in \mathbb{P}_{n-1+k}$. Therefore, by (1.1), $E_n(\gamma_n p) = 0$. Hence,

$$\int_{a}^{b} \gamma_n(x)p(x)w(x) \, dx = \sum_{j=1}^{n} w_j \gamma_n(x_j)p(x_j) = 0,$$

where we use the fact that $\gamma_n(x_j) = 0$, for $j = 1, \ldots, n$. 

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Conversely, suppose (a) and (b) hold. To show this implies (1.1), we proceed as follows. Let \( p \in \mathbb{P}_{n-1+k} \), and divide \( p \) by \( \gamma_n \):

\[
p(x) = q(x)\gamma_n(x) + r(x), \quad x \in [a, b],
\]

where \( q \in \mathbb{P}_{k-1} \) is the quotient and \( r \in \mathbb{P}_{n-1} \) is the remainder. We note,

\[
\int_a^b p(x)w(x)\, dx = \int_a^b q(x)\gamma_n(x)w(x)\, dx + \int_a^b r(x)w(x)\, dx.
\]

Now, since \( q \in \mathbb{P}_{k-1} \), the first term vanishes by (b). That is,

\[
\int_a^b p(x)w(x)\, dx = \int_a^b r(x)w(x)\, dx. \tag{1.2}
\]

Next, note that since \( r \in \mathbb{P}_{n-1} \), (a) implies

\[
\int_a^b r(x)w(x)\, dx = \sum_{j=1}^n w_jr(x_j) = \sum_{j=1}^n w_j(p(x_j) - q(x_j)\gamma_n(x_j)) = \sum_{j=1}^n w_jp(x_j), \tag{1.3}
\]

where we again used \( \gamma_n(x_j) = 0 \) for \( j = 1, \ldots, n \). Combining (1.2) and (1.3), we get

\[
\int_a^b p(x)w(x)\, dx = \sum_{j=1}^n w_jp(x_j).
\]

That is, \( E_n(p) = 0 \). Since \( p \in \mathbb{P}_{n-1+k} \) was arbitrary, this finishes the proof. \( \square \)

**Remark 1.2.** We make the following comments:

1. In the above theorem \( k = n \) is optimal and leads to a quadrature formula with optimal degree of exactness \( 2n - 1 \). This is the \( n \)-point Gaussian quadrature formula corresponding to the weight function \( w \).

2. Observe that by (b) in the theorem, and with \( k = n \),

\[
(\gamma_n, p)_w = 0, \quad \forall p \in \mathbb{P}_{n-1}, \tag{1.4}
\]

where \((\cdot, \cdot)_w\) is (weighted) \( L^2_w \) inner product:

\[
(f, g)_w = \int_a^b f(x)g(x)w(x)\, dx, \quad f, g \in L^2_w[a, b].
\]

That is, (1.4) says \( \gamma_n \), which is an \( n \)th degree monic polynomial, is orthogonal to every \( p \in \mathbb{P}_{n-1} \). Therefore, \( \gamma_n = \pi_n \) with \( \pi_n \) the degree \( n \) (monic) orthogonal polynomial with respect to \((\cdot, \cdot)_w\). This also shows that the nodes of the \( n \)-point Gaussian quadrature formula, with the weight function \( w \), are the roots of the degree \( n \) orthogonal polynomial \( \pi_n \).

## 2 Convergence of Gaussian quadrature

The discussion here is focused on proving convergence of a Gaussian quadrature formula on a closed and bounded interval. First we record the following important result [1].

**Lemma 2.1.** The weights of a Gaussian quadrature formula are positive.
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**Proof.** Let $I_n(f) = \sum_{j=1}^{n} w_j f(x_j)$, be an $n$-point Gaussian quadrature formula for approximating $I(f) = \int_a^b f(x)w(x) \, dx$. Let $\ell_i$ be the $i$th elementary Lagrange polynomial,

$$\ell_i(x) = \prod_{k=1}^{n} \frac{x - x_k}{x_i - x_k}.$$ 

Note that $\ell_i \in \mathbb{P}_{n-1}$ and so $\ell_i^2 \in \mathbb{P}_{2n-2}$. Therefore, since $I_n$ has degree of exactness $2n - 1$, we have

$$0 < \int_a^b \ell_i(x)^2 w(x) \, dx = \sum_{j=1}^{n} w_j \ell_i(x_j)^2 = w_i, \quad i = 1, \ldots, n. \quad \Box$$

Consider a closed and bounded interval $[a, b]$. Recall that for a given $f \in C[a, b]$, there exists a unique polynomial $q_n^*$ of degree $\leq n$, for which $\rho_n(f) = \|f - q_n^*\|_{\infty}$, where $\rho_n(f) = \inf_{q \in \mathbb{P}_n} \|f - q\|_{\infty}$ is the minimax error. We record the following basic fact:

**Lemma 2.2.** Let $[a, b]$ be a closed and bounded interval. For $f \in C[a, b]$, $\rho_n(f) \to 0$, as $n \to \infty$.

**Proof.** This follows easily from Weierstrass Approximation Theorem. \hfill \Box

The following is an important result for Gaussian quadrature formulas. The proof presented below is a standard argument.

**Theorem 2.3.** Let $[a, b]$ be a closed and bounded interval. Let $I(f) = \int_a^b f(x)w(x) \, dx$ where $w \geq 0$ is a weight function, and let $I_n(f) = \sum_{j=1}^{n} w_j f(x_j)$ be an $n$-point Gaussian quadrature formula corresponding to the weight function $w$. Then, the following hold:

(a) The error $E_n(f) = I(f) - I_n(f)$ satisfies

$$|E_n(f)| \leq 2\rho_{2n-1}(f) \int_a^b w(x) \, dx, \quad \forall f \in C[a, b].$$

(b) $\lim_{n \to \infty} I_n(f) = I(f)$, for all $f \in C[a, b]$.

**Proof.** Let $f \in C[a, b]$, and let $p_{2n-1}$ be its minimax approximation in $\mathbb{P}_{2n-1}$. We know that $I_n(p_{2n-1}) = I(p_{2n-1})$. Note that,

$$|E_n(f)| = |I(f) - I_n(p_{2n-1}) + I_n(p_{2n-1}) - I_n(f)|$$

$$\leq |I(f) - I(p_{2n-1})| + |I_n(p_{2n-1}) - I_n(f)|$$

$$\leq \int_a^b |f(x) - p_{2n-1}(x)|w(x) \, dx + \sum_{j=1}^{n} w_j |p_{2n-1}(x_j) - f(x)| \quad \text{(recall $w_j > 0$ by Lemma 2.1)}$$

$$\leq \rho_{2n-1}(f) \int_a^b w(x) \, dx + \sum_{j=1}^{n} w_j \rho_{2n-1}(f) = 2\rho_{2n-1}(f) \int_a^b w(x) \, dx,$$

where we also used $\sum_{j=1}^{n} w_j = \int_a^b w(x) \, dx$. This establishes statement (a) of the theorem. The statement (b) follows from (a) and Lemma 2.2. \hfill \Box

**Remark 2.4.** The above result shows an interesting property of Gaussian quadrature. Note that the speed at which $\rho_n(f)$ converges to zero increases with the smoothness of the $f$; see e.g., [2, Section 4.6–4.7]. The above theorem shows that Gaussian quadrature formulas inherit this property. That is, Gaussian quadrature takes advantage of additional smoothness in the integrand. This is in contrast to most composite rules.
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References