A brief note on Gaussian quadrature

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Abstract

We review some key results regarding Gaussian quadrature.

1 A key result

Let $I_n(f)$ be an n-point quadrature formula

$$I_n(f) = \sum_{j=1}^{n} w_j f(x_j),$$

where x_j are quadrature nodes, and w_j are quadrature weights. The rule $I_n(f)$ is used to approximate $I(f) = \int_a^b f(x)w(x)\,dx$, where $w \geq 0$ is a weight function. We denote bt $E_n(f) = I(f) - I_n(f)$ the quadrature error. The following result from [1] reveals some connections between quadrature and ideas from approximation theory. The argument presented here follows in similar lines as that in [1].

Theorem 1.1. *Let $I_n(f) = \sum_{j=1}^{n} w_j f(x_j)$ be an n-point quadrature formula for approximating $I(f) = \int_a^b f(x)w(x)\,dx$, where $w \geq 0$ is a weight function. Define,*

$$\gamma_n(x) = \prod_{k=1}^{n} (x - x_k).$$

Given $0 \leq k \leq n$, we have

$$E_n(p) = 0, \quad \forall p \in \mathbb{P}_{n-1+k}, \tag{1.1}$$

if and only if the following hold:

(a) I_n *is interpolatory.*

(b) $\int_a^b \gamma_n(x)p(x)w(x)\,dx = 0, \quad \forall p \in \mathbb{P}_{k-1}.$

Proof. Suppose (1.1) holds. This immediately implies (a). To see (b), notice that for $p \in \mathbb{P}_{k-1}$, $\gamma_n p \in \mathbb{P}_{n-1+k}$. Therefore, by (1.1), $E_n(\gamma_n p) = 0$. Hence,

$$\int_a^b \gamma_n(x)p(x)w(x)\,dx = \sum_{j=1}^{n} w_j \gamma_n(x_j)p(x_j) = 0,$$

where we use the fact that $\gamma_n(x_j) = 0$, for $j = 1, \ldots, n$.

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Conversely, suppose (a) and (b) hold. To show this implies (1.1), we proceed as follows. Let $p \in \mathbb{P}_{n-1+k}$, and divide p by γ_n:

$$p(x) = q(x)\gamma_n(x) + r(x), \quad x \in [a,b],$$

where $q \in \mathbb{P}_{k-1}$ is the quotient and $r \in \mathbb{P}_{n-1}$ is the remainder. We note,

$$\int_a^b p(x)w(x)\,dx = \int_a^b q(x)\gamma_n(x)w(x)\,dx + \int_a^b r(x)w(x)\,dx.$$

Now, since $q \in \mathbb{P}_{k-1}$, the first term vanishes by (b). That is,

$$\int_a^b p(x)w(x)\,dx = \int_a^b r(x)w(x)\,dx. \tag{1.2}$$

Next, note that since $r \in \mathbb{P}_{n-1}$, (a) implies

$$\int_a^b r(x)w(x)\,dx = \sum_{j=1}^n w_j r(x_j) = \sum_{j=1}^n w_j(p(x_j) - q(x_j)\gamma_n(x_j)) = \sum_{j=1}^n w_j p(x_j), \tag{1.3}$$

where we again used $\gamma_n(x_j) = 0$ for $j = 1, \ldots, n$. Combining (1.2) and (1.3), we get

$$\int_a^b p(x)w(x)\,dx = \sum_{j=1}^n w_j p(x_j).$$

That is, $E_n(p) = 0$. Since $p \in \mathbb{P}_{n-1+k}$ was arbitrary, this finishes the proof. $\square$

Remark 1.2. *We make the following comments:*

1. *In the above theorem $k = n$ is optimal and leads to a quadrature formula with optimal degree of exactness $2n-1$. This is the n-point Gaussian quadrature formula corresponding to the weight function w.*

2. *Observe that by (b) in the theorem, and with $k = n$,*

$$(\gamma_n, p)_w = 0, \quad \forall p \in \mathbb{P}_{n-1}, \tag{1.4}$$

where $(\cdot,\cdot)_w$ is (weighted) L^2_w inner product:

$$(f,g)_w = \int_a^b f(x)g(x)w(x)\,dx, \quad f,g \in L^2_w[a,b].$$

That is, (1.4) *says γ_n, which is an nth degree monic polynomial, is orthogonal to every $p \in \mathbb{P}_{n-1}$. Therefore, $\gamma_n = \pi_n$ with π_n the degree n (monic) orthogonal polynomial with respect to $(\cdot,\cdot)_w$. This also shows that the nodes of the n-point Gaussian quadrature formula, with the weight function w, are the roots of the degree n orthogonal polynomial π_n.*

2 Convergence of Gaussian quadrature

The discussion here is focused on proving convergence of a Gaussian quadrature formula on a closed and bounded interval. First we record the following important result [1].

Lemma 2.1. *The weights of a Gaussian quadrature formula are positive.*

Proof. Let $I_n(f) = \sum_{j=1}^n w_j f(x_j)$, be an n-point Gaussian quadrature formula for approximating $I(f) = \int_a^b f(x)w(x)\,dx$. Let ℓ_i be the ith elementary Lagrange polynomial,

$$\ell_i(x) = \prod_{\substack{k=1\\k\neq i}}^{n} \frac{x - x_k}{x_i - x_k}.$$

Note that $\ell_i \in \mathbb{P}_{n-1}$ and so $\ell_i^2 \in \mathbb{P}_{2n-2}$. Therefore, since I_n has degree of exactness $2n-1$, we have

$$0 < \int_a^b \ell_i(x)^2 w(x)\,dx = \sum_{j=1}^n w_j \ell_i(x_j)^2 = w_i, \quad i = 1, \ldots, n. \qquad \square$$

Consider a closed and bounded interval $[a, b]$. Recall that for a given $f \in C[a, b]$, there exists a unique polynomial q_n^* of degree $\le n$, for which $\rho_n(f) = \|f - q_n^*\|_\infty$, where $\rho_n(f) = \inf_{q\in\mathbb{P}_n} \|f - q\|_\infty$ is the minimax error. We record the following basic fact:

Lemma 2.2. *Let $[a, b]$ be a closed and bounded interval. For $f \in C[a, b]$, $\rho_n(f) \to 0$, as $n \to \infty$.*

Proof. This follows easily from Weierstrass Approximation Theorem. □

The following is an important result for Gaussian quadrature formulas. The proof presented below is a standard argument.

Theorem 2.3. *Let $[a, b]$ be a closed and bounded interval. Let $I(f) = \int_a^b f(x)w(x)\,dx$ where $w \ge 0$ is a weight function, and let $I_n(f) = \sum_{j=1}^n w_j f(x_j)$ be an n-point Gaussian quadrature formula corresponding to the weight function w. Then, the following hold:*

(a) The error $E_n(f) = I(f) - I_n(f)$ satisfies

$$|E_n(f)| \le 2\rho_{2n-1}(f) \int_a^b w(x)\,dx, \quad \forall f \in C[a, b].$$

(b) $\lim_{n\to\infty} I_n(f) = I(f)$, for all $f \in C[a, b]$.

Proof. Let $f \in C[a, b]$, and let p_{2n-1}^* be its minimax approximation in $\mathbb{P}_{2n-1}$. We know that $I_n(p_{2n-1}^*) = I(p_{2n-1}^*)$. Note that,

$$\begin{aligned}
|E_n(f)| &= |I(f) - I_n(p_{2n-1}^*) + I_n(p_{2n-1}^*) - I_n(f)| \\
&\le |I(f) - I(p_{2n-1}^*)| + |I_n(p_{2n-1}^*) - I_n(f)| \\
&\le \int_a^b |f(x) - p_{2n-1}^*(x)| w(x)\,dx + \sum_{j=1}^n w_j |p_{2n-1}^*(x_j) - f(x_j)| \qquad \text{(recall } w_j > 0 \text{ by Lemma 2.1)} \\
&\le \rho_{2n-1}(f) \int_a^b w(x)\,dx + \left(\sum_{j=1}^n w_j\right) \rho_{2n-1}(f) = 2\rho_{2n-1}(f) \int_a^b w(x)\,dx,
\end{aligned}$$

where we also used $\sum_{j=1}^n w_j = \int_a^b w(x)\,dx$. This establishes statement (a) of the theorem. The statement (b) follows from (a) and Lemma 2.2. □

Remark 2.4. *The above result shows an interesting property of Gaussian quadrature. Note that the speed at which $\rho_n(f)$ converges to zero increases with the smoothness of the f; see e.g., [2, Section 4.6–4.7]. The above theorem shows that Gaussian quadrature formulas inherit this property. That is, Gaussian quadrature takes advantage of additional smoothness in the integrand. This is in contrast to most composite rules.*

References

[1] Gautschi, Walter. Numerical analysis. Springer Science & Business Media, 2011.

[2] Kendall E. Atkinson. *An introduction to numerical analysis*. John Wiley & Sons Inc., New York, second edition, 1989.