Dependence of simple eigenpairs to differentiable perturbations

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Abstract

Let $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be a differentiable matrix valued function. Suppose $\mathbf{A}(\mathbf{0})$ has a real simple eigenvalue λ_0. We show here that under for $\boldsymbol{\xi}$ sufficiently close to the origin, $\mathbf{A}(\boldsymbol{\xi})$ has a real eigenvalue $\lambda(\boldsymbol{\xi})$, and that $\lambda(\boldsymbol{\xi})$ depends differentiably on $\boldsymbol{\xi}$. A related result regarding the eigenvectors is also reviewed.

1 Introduction

Consider a matrix valued function $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $\mathbf{A} = \mathbf{A}(\boldsymbol{\xi})$ with $\boldsymbol{\xi} \in \mathbb{R}^n$. Here we show that if $\mathbf{A}(\mathbf{0})$ has a *real simple* eigenvalue λ_0 and that $\mathbf{A}$ depends differentiably on $\boldsymbol{\xi}$, then there exists a neighborhood $U \subset \mathbb{R}^n$ of $\boldsymbol{\xi} = \mathbf{0}$, such that for each $\boldsymbol{\xi} \in U$, $\mathbf{A}(\boldsymbol{\xi})$ has a *real* eigenvalue $\lambda(\boldsymbol{\xi})$ which depends differentiably on $\boldsymbol{\xi}$. This result is not new; the idea of the proof presented here is mainly based on a result in [1] with a slight modification of considering $\mathbb{R}$ as the base field. Moreover, under the same assumption, it can be shown that we have an eigenvector $\boldsymbol{v}(\boldsymbol{\xi}) \in \mathbb{R}^n$ for each $\boldsymbol{\xi} \in U$ corresponding to $\lambda(\boldsymbol{\xi})$, and $\boldsymbol{v}$ depends differentiably on $\boldsymbol{\xi}$ in U.

2 Implicit Function Theorem

Here we state the following special case of the Implicit Function Theorem (see e.g. [3, 2]) which is used in this note.

Theorem 2.1. *Let $D \subset \mathbb{R}^n \times \mathbb{R}$ be an open set and $F : D \to \mathbb{R}$ a C^1 function. Assume that $(\boldsymbol{x}_0, y_0) \in D$ is such that*

$$F(\boldsymbol{x}_0, y_0) = 0, \quad \text{and} \quad \frac{\partial}{\partial y} F(\boldsymbol{x}_0, y)\Big|_{y=y_0} \neq 0.$$

Then, there exists an open set $U \subset \mathbb{R}^n$ containing $\boldsymbol{x}_0$ and an open set $V \subset \mathbb{R}$ containing y_0 with $U \times V \subset D$, and a unique C^1 function $h : U \to V$ such that,

$$h(\boldsymbol{x}_0) = y_0 \quad \text{and} \quad F(\boldsymbol{x}, h(\boldsymbol{x})) = 0, \forall \boldsymbol{x} \in U.$$

3 Main results

Here we present the main results we are concerned with in this note. The idea behind the proof of the theorem below is from [1]. The only difference here is that we

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restrict the base field to reals and consider real eigenvalues.

Theorem 3.1. *Let* $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ *be a differentiable matrix valued function. Suppose* $\mathbf{A}(\mathbf{0})$ *has a real simple eigenvalue* λ_0*. Then, there exists a neighborhood* $U \subset \mathbb{R}^n$ *of the origin, such that for each* $\boldsymbol{\xi} \in U$*,* $\mathbf{A}(\boldsymbol{\xi})$ *has a real eigenvalue* $\lambda(\boldsymbol{\xi})$*. Moreover,* $\lambda(\boldsymbol{\xi})$ *depends differentiably on* $\boldsymbol{\xi}$*, and* $\lambda(\mathbf{0}) = \lambda_0$*.*

Proof. Consider the function $p : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ given by,

$$p(\boldsymbol{\xi}, y) = \det(\mathbf{A}(\boldsymbol{\xi}) - y\mathbf{I}).$$

Note that by the assumptions on λ_0, we have $p(\mathbf{0}, \lambda_0) = 0$. Moreover, we note that $p(\mathbf{0}, y) = (y - \lambda_0)r(y)$, with $r(y)$ a polynomial of degree $n - 1$. Since λ_0 is a simple eigenvalue, i.e. a simple root of $p(\mathbf{0}, y)$, it follows that $\frac{\partial}{\partial y} p(\mathbf{0}, y) \mid_{y=\lambda_0} = r(\lambda_0) \neq 0$.

Thus, we have the following:

$$p(\mathbf{0}, \lambda_0) = 0, \quad \frac{\partial p}{\partial y}(\mathbf{0}, y)\Big|_{y=\lambda_0} \neq 0.$$

Therefore, by the Implicit Function Theorem, there exists a neighborhood $U \subset \mathbb{R}^n$ of $\mathbf{0}$, a neighborhood $V \subset \mathbb{R}$ of λ_0, and a C^1 function $\lambda : U \to V$ such that $\lambda(\mathbf{0}) = \lambda_0$ and $p(\boldsymbol{\xi}, \lambda(\boldsymbol{\xi})) = 0$ for every $\boldsymbol{\xi} \in U$. That is

$$\det(\mathbf{A}(\boldsymbol{\xi}) - \lambda(\boldsymbol{\xi})\mathbf{I}) = 0, \quad \forall\, \boldsymbol{\xi} \in U,$$

so that $\lambda(\boldsymbol{\xi})$ is an eigenvalue of $\mathbf{A}(\boldsymbol{\xi})$ with $\lambda(\boldsymbol{\xi}) \in V \subset \mathbb{R}$. □

The following result is a consequence of Theorem 8 in [1, p. 102]:

Theorem 3.2. *Let* $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ *be a differentiable matrix valued function. Suppose* $\mathbf{A}(\mathbf{0})$ *has a real simple eigenvalue* λ_0*. Then, for* $\boldsymbol{\xi}$ *in a neighborhood of the origin, we can choose an eigenvector* $\boldsymbol{v}(\boldsymbol{\xi}) \in \mathbb{R}^n$ *of* $\mathbf{A}(\boldsymbol{\xi})$ *corresponding to* $\lambda(\boldsymbol{\xi})$*, with* $\lambda(\boldsymbol{\xi}) \in \mathbb{R}$ *as in Theorem 3.1, to depend differentiably on* $\boldsymbol{\xi}$*.*

In numerical computations, when using an eigensolver for computing eigenvalues and eigenvectors of $\mathbf{A}(\boldsymbol{\xi})$ for a sequence $\{\boldsymbol{\xi}_n\}$, in a neighborhood of $\mathbf{0}$, with $\boldsymbol{\xi}_n \to \mathbf{0}$, the sign of the computed eigenvectors might flip between the iterations. Aligning the eigenvectors might become necessary for numerical tests. This can be done easily if for each $\boldsymbol{\xi}$, we compute a unit eigenvector $\boldsymbol{v}(\boldsymbol{\xi})$, corresponding to $\lambda(\boldsymbol{\xi})$, and replace $\boldsymbol{v}(\boldsymbol{\xi})$ by $\tilde{\boldsymbol{v}}(\boldsymbol{\xi}) = \operatorname{sign}\big(\boldsymbol{v}(\mathbf{0})^\top \boldsymbol{v}(\boldsymbol{\xi})\big)\boldsymbol{v}(\boldsymbol{\xi})$. We call this the *positively oriented* eigenvector corresponding to $\lambda(\boldsymbol{\xi})$, motivated by the fact that $\boldsymbol{v}(0)^\top \tilde{\boldsymbol{v}}(\boldsymbol{\xi}) > 0$ (as long as $\boldsymbol{v}(\boldsymbol{\xi})$ does not become perpendicular to $\boldsymbol{v}(\mathbf{0})$).

References

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[2] Jerrold E. Marsden and Michael J. Hoffman. *Elementary classical analysis*. W.H. Freeman and Company, 2nd edition, 1993.

[3] Walter Rudin. *Principles of Mathematical Analysis*. McGraw-Hill Book Company, 2nd edition, 1964.