

Dependence of simple eigenpairs to differentiable perturbations

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Abstract

Let $\mathbf{A} : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ be a differentiable matrix valued function. Suppose $\mathbf{A}(\mathbf{0})$ has a real simple eigenvalue λ_0 . We show here that for $\boldsymbol{\xi}$ sufficiently close to the origin, $\mathbf{A}(\boldsymbol{\xi})$ has a real eigenvalue $\lambda(\boldsymbol{\xi})$, and that $\lambda(\boldsymbol{\xi})$ depends differentiably on $\boldsymbol{\xi}$. A related result regarding the eigenvectors is also reviewed.

1 Introduction

Consider a matrix valued function $\mathbf{A} : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$, $\mathbf{A} = \mathbf{A}(\boldsymbol{\xi})$ with $\boldsymbol{\xi} \in \mathbb{R}^n$. Here we show that if $\mathbf{A}(\mathbf{0})$ has a *real simple* eigenvalue λ_0 and that \mathbf{A} depends differentiably on $\boldsymbol{\xi}$, then there exists a neighborhood $U \subset \mathbb{R}^n$ of $\boldsymbol{\xi} = \mathbf{0}$, such that for each $\boldsymbol{\xi} \in U$, $\mathbf{A}(\boldsymbol{\xi})$ has a *real* eigenvalue $\lambda(\boldsymbol{\xi})$ which depends differentiably on $\boldsymbol{\xi}$. This result is not new; the idea of the proof presented here is mainly based on a result in [1] with a slight modification of considering \mathbb{R} as the base field. Moreover, under the same assumption, it can be shown that we have an eigenvector $\mathbf{v}(\boldsymbol{\xi}) \in \mathbb{R}^n$ for each $\boldsymbol{\xi} \in U$ corresponding to $\lambda(\boldsymbol{\xi})$, and \mathbf{v} depends differentiably on $\boldsymbol{\xi}$ in U .

In what follows, we need the following special case of the Implicit Function Theorem.¹

¹ See, e.g. [3, 2].

Theorem 1.1. *Let $D \subset \mathbb{R}^n \times \mathbb{R}$ be an open set and $F : D \rightarrow \mathbb{R}$ a C^1 function. Assume that $(\mathbf{x}_0, y_0) \in D$ is such that*

$$F(\mathbf{x}_0, y_0) = 0, \quad \text{and} \quad \left. \frac{\partial}{\partial y} F(\mathbf{x}_0, y) \right|_{y=y_0} \neq 0.$$

Then, there exists an open set $U \subset \mathbb{R}^n$ containing \mathbf{x}_0 and an open set $V \subset \mathbb{R}$ containing y_0 with $U \times V \subset D$, and a unique C^1 function $h : U \rightarrow V$ such that,

$$h(\mathbf{x}_0) = y_0 \quad \text{and} \quad F(\mathbf{x}, h(\mathbf{x})) = 0, \text{ for all } \mathbf{x} \in U.$$

2 Differentiability of simple eigenpairs

First, we consider a key result regarding differentiability of sample eigenvalues. The idea behind the proof of the theorem below is from [1]. However, here, we restrict the base field to real numbers and consider real eigenvalues.

Theorem 2.1. *Let $\mathbf{A} : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ be a differentiable matrix valued function. Suppose $\mathbf{A}(\mathbf{0})$ has a real simple eigenvalue λ_0 . Then, there exists a neighborhood $U \subset \mathbb{R}^n$ of the origin, such that for each $\boldsymbol{\xi} \in U$, $\mathbf{A}(\boldsymbol{\xi})$ has a real eigenvalue $\lambda(\boldsymbol{\xi})$. Moreover, $\lambda(\boldsymbol{\xi})$ depends differentiably on $\boldsymbol{\xi}$, and $\lambda(\mathbf{0}) = \lambda_0$.*

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Last revised: October 24, 2024.

Proof. Consider the function $p : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ given by,

$$p(\boldsymbol{\xi}, y) = \det(\mathbf{A}(\boldsymbol{\xi}) - y\mathbf{I}).$$

Note that by the assumptions on λ_0 , we have $p(\mathbf{0}, \lambda_0) = 0$. Moreover, we note that $p(\mathbf{0}, y) = (y - \lambda_0)r(y)$, with $r(y)$ a polynomial of degree $n - 1$. Since λ_0 is a simple eigenvalue, i.e. a simple root of $p(\mathbf{0}, y)$, it follows that $\frac{\partial}{\partial y}p(\mathbf{0}, y)|_{y=\lambda_0} = r(\lambda_0) \neq 0$. Hence, we have

$$p(\mathbf{0}, \lambda_0) = 0 \quad \text{and} \quad \frac{\partial p}{\partial y}(\mathbf{0}, y)|_{y=\lambda_0} \neq 0.$$

Therefore, by the Implicit Function Theorem, there exists a neighborhood $U \subset \mathbb{R}^n$ of $\mathbf{0}$, a neighborhood $V \subset \mathbb{R}$ of λ_0 , and a C^1 function $\lambda : U \rightarrow V$ such that $\lambda(\mathbf{0}) = \lambda_0$ and $p(\boldsymbol{\xi}, \lambda(\boldsymbol{\xi})) = 0$ for every $\boldsymbol{\xi} \in U$. That is

$$\det(\mathbf{A}(\boldsymbol{\xi}) - \lambda(\boldsymbol{\xi})\mathbf{I}) = 0, \quad \text{for all } \boldsymbol{\xi} \in U,$$

so that $\lambda(\boldsymbol{\xi})$ is an eigenvalue of $\mathbf{A}(\boldsymbol{\xi})$ with $\lambda(\boldsymbol{\xi}) \in V \subset \mathbb{R}$. □

The following result is a consequence of Theorem 8 in [1, p. 102]:

Theorem 2.2. *Let $\mathbf{A} : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ be a differentiable matrix valued function. Suppose $\mathbf{A}(\mathbf{0})$ has a real simple eigenvalue λ_0 . Then, for $\boldsymbol{\xi}$ in a neighborhood of the origin, we can choose an eigenvector $\mathbf{v}(\boldsymbol{\xi}) \in \mathbb{R}^n$ of $\mathbf{A}(\boldsymbol{\xi})$ corresponding to $\lambda(\boldsymbol{\xi})$, with $\lambda(\boldsymbol{\xi}) \in \mathbb{R}$ as in Theorem 2.1, to depend differentiably on $\boldsymbol{\xi}$.*

In numerical computations, when using an eigensolver for computing eigenvalues and eigenvectors of $\mathbf{A}(\boldsymbol{\xi})$ for a sequence $\{\boldsymbol{\xi}_n\}$, in a neighborhood of $\mathbf{0}$, with $\boldsymbol{\xi}_n \rightarrow \mathbf{0}$, the sign of the computed eigenvectors might flip between with different choice of n . Aligning the eigenvectors might become necessary for numerical tests.² This can be done easily if for each $\boldsymbol{\xi}$, we compute a unit eigenvector $\mathbf{v}(\boldsymbol{\xi})$, corresponding to $\lambda(\boldsymbol{\xi})$, and replace $\mathbf{v}(\boldsymbol{\xi})$ by $\tilde{\mathbf{v}}(\boldsymbol{\xi}) = \text{sign}(\mathbf{v}(\mathbf{0})^\top \mathbf{v}(\boldsymbol{\xi}))\mathbf{v}(\boldsymbol{\xi})$. We call this the *positively oriented* eigenvector corresponding to $\lambda(\boldsymbol{\xi})$, motivated by the fact that $\mathbf{v}(\mathbf{0})^\top \tilde{\mathbf{v}}(\boldsymbol{\xi}) > 0$. This is the case as long as $\mathbf{v}(\boldsymbol{\xi})$ does not become perpendicular to $\mathbf{v}(\mathbf{0})$.

² *This is needed, for example, if one aims to examine the change in the eigenvector $\mathbf{v}(\boldsymbol{\xi})$ for small perturbations of $\boldsymbol{\xi}$.*

References

- [1] Peter D. Lax. *Linear Algebra*. Wiley-Interscience, 1996.
- [2] Jerrold E. Marsden and Michael J. Hoffman. *Elementary classical analysis*. W.H. Freeman and Company, 2nd edition, 1993.
- [3] Walter Rudin. *Principles of Mathematical Analysis*. McGraw-Hill Book Company, 2nd edition, 1964.