On compact operators

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Abstract

In this basic note, we consider some basic properties of compact operators. We also consider the spectrum of compact operators on Hilbert spaces. A basic numerical example involving a compact integral operator is provided for further illustration.

1 Introduction

The goal of this brief note is to collect some of the basic properties of compact operators on normed linear spaces. The results discussed here are all standard and can be found in standard references such as [4, 9, 8, 7, 2, 5] to name a few. The point of this note is to provide an accessible introduction to some basic properties of compact operators. After stating some preliminaries and basic definitions in Section 2, we follow up by discussing some examples of compact operators in Section 3. Then, in Section 4, we discuss the range space of a compact operator, where we will see that the range of a compact opeator is "almost finite-dimensional". In Section 5, we discuss a basic result on approximation of compact operators by finite-dimensional operators. Section 6 recalls some basic facts regarding the spectrum of linear operators on Banach spaces. Sections 7–9 are concerned with spectral properties of compact operators on Hilbert spaces and Fredholm's theorem of alternative. Finally a numerical example is presented in Section 11.

2 Preliminaries

Let us begin by recalling the notion of precompact and relatively compact sets.

Definition 2.1. *(Relatively Compact)*
Let X be a metric space; $A \subseteq X$ is relatively compact in X, if $\bar{A}$ is compact in X.

Definition 2.2. *(Precompact)*
Let X be a metric space; $A \subseteq X$ is precompact (also called totally bounded) if for every $\epsilon > 0$, there exist finitely many points $x_1, \ldots, x_N$ in A such that $\cup_1^N B(x_i, \epsilon)$ covers A.

The following Theorem shows that when we are working in a complete metric space, precompactness and relative compactness are equivalent.

Theorem 2.3. *Let X be a metric space. If $A \subseteq X$ is relatively compact then it is precompact. Moreover, if X is complete then the converse holds also.*

Then, we define a compact operator as below.

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Definition 2.4. *Let X and Y be two normed linear spaces and $T : X \to Y$ a linear map between X and Y. T is called a compact operator if for all bounded sets $E \subseteq X$, $T(E)$ is relatively compact in Y.*

By Definition 2.4, if $E \subset X$ is a bounded set, then $\overline{T(E)}$ is compact in Y. The following basic result shows a couple of different ways of looking at compact operators.

Theorem 2.5. *Let X and Y be two normed linear spaces; suppose $T : X \to Y$, is a linear operator. Then the following are equivalent.*

1. *T is compact.*
2. *The image of the open unit ball under T is relatively compact in Y.*
3. *For any bounded sequence $\{x_n\}$ in X, there exist a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ that converges in Y.*

Let us denote by $B[X]$ the set of all bounded linear operators on a normed linear space space X:

$$B[X] = \{T : X \to X|\ T \text{ is a bounded linear transformation.}\}.$$

Note that equipped by the operator norm $B[X]$ is a normed linear space. It is simple to show that compact operators form a subspace of $B[X]$. The following result (cf. [8] for a proof) shows that the set of compact operators is in fact a closed subspace of $B[X]$.

Theorem 2.6. *Let $\{T_n\}$ be a sequence of compact operators on a normed linear space X. Suppose $T_n \to T$ in $B[X]$. Then, T is also a compact operator.*

Another interesting fact regarding compact linear operators is that they form an ideal of the ring of bounded linear mappings $B[X]$; this follows from the following result.

Lemma 2.7. *Let X be a normed linear space, and let T and S be in $B[X]$. If T is compact, then so are ST and TS.*

Proof. Consider the mapping ST. Let $\{x_n\}$ be a bounded sequence in X. Then, by Theorem 2.5(3), there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ that converges in X:

$$Tx_{n_k} \to y^* \in X.$$

Now, since S is continuous, it follows that $STx_{n_k} \to S(y^*)$; that is, $\{STx_{n_k}\}$ converges in X also, and so ST is compact. To show TS is compact, take a bounded sequence $\{x_n\}$ in X and note that $\{Sx_n\}$ is bounded also (since S is continuous). Thus, again by Theorem 2.5(3), there exists a subsequence $\{TSx_{n_k}\}$ which converges in X, and thus, TS is also compact. □

Remark 2.8. *A compact linear operator of an infinite dimensional normed linear space is not invertible in $B[X]$. To see this, suppose that T has an inverse S in $B[X]$. Now, applying the previous Lemma, we get that $I = TS = ST$ is also compact. However, this implies that the closed unit ball in X is compact, which is not possible since we assumed X is infinite dimensional[1].*

3 Some examples of compact operators

Here we consider two special instances of compact operators: the finite-dimensional (or finite-rank) operators, and the Hilbert-Schmidt operators.

[1] Recall that the closed unit ball in a normed linear space X is compact if and only if X is finite dimensional.

Finite-dimensional operators Let $T : X \to Y$ be a continuous linear mapping between normed linear spaces If the range space $\mathsf{Ran}(T)$ is of finite dimension, $dim(\mathsf{Ran}(T)) < \infty$, we call T a *finite-dimensional* operator. It is straightforward to see that finite-dimensional operators are compact. This is seen by noting that for a bounded set $E \subseteq X$, $\overline{T(E)}$ is closed and bounded in the finite-dimensional subspace $\mathsf{Ran}(T) \subseteq Y$. Therefore, Heine-Borel Theorem applies, and $\overline{T(E)}$ is compact in $\mathsf{Ran}(T) \subseteq Y$.

Hilbert-Schmidt operators Let $D \subset \mathbb{R}^n$ be a bounded domain. We call a function $k : D \times D \to \mathbb{R}$ a Hilbert-Schmidt kernel if

$$\int_D \int_D |k(x, y)|^2 \, dx \, dy < \infty,$$

that is, $k \in L^2(D \times D)$ (note that one special case is when k is a continuous function on $D \times D$). Define the integral operator K on $L^2(D)$, $K : u \to Ku$ for $u \in L^2(D)$, by

$$[Ku](x) = \int_D k(x, y)u(y) \, dy. \tag{3.1}$$

Clearly, K is linear; moreover, it is simple to show that $K : L^2(D) \to L^2(D)$:

Let $u \in L^2(D)$, then

$$\begin{aligned} \int_D \Big|(Ku)(x)\Big|^2 dx &= \int_D \Big| \int_D k(x, y)u(y) \, dy \Big|^2 dx \\ &\leq \int_D \Big(\int_D |k(x, y)|^2 \, dy \Big) \Big(\int_D |u(y)|^2 \, dy \Big) \, dx \qquad \text{(Cauchy-Schwarz)} \\ &= ||k||_{L^2(D \times D)} ||u||_{L^2(D)} < \infty, \end{aligned}$$

so that $Ku \in L^2(D)$. The mapping K is what we call a *Hilbert-Schmidt operator*.

Lemma 3.1. *Let D be a bounded domain in R^n and let $k \in L^2(D \times D)$ be a Hilbert-Schmidt kernel. Then, the integral operator $K : L^2(D) \to L^2(D)$ given by $[Ku](x) = \int_D k(x, y)u(y) \, dy$ is a compact operator.*

The basic idea of the proof is to write the operator K as a limit of finite-dimensional operators and then apply Theorem 2.6.

Remark 3.2. *One can think of Hilbert-Schmidt operators as generalizations of the idea of matrices to infinite-dimensional spaces. Note that if A is an $n \times n$ matrix (a linear mapping on $\mathbb{R}^n$), then, the action Ax of A on a vector $x \in \mathbb{R}^n$ is given by*

$$[Ax]_i = \sum_{j=1}^{n} A_{ij} x_j \tag{3.2}$$

Now note that,

$$[Ku](x) = \int_D k(x, y)u(y) \, dy$$

is an analog of (3.2) *with the summation replaced with an integral.*

4 Range of a compact operator

We saw in the previous section that a continuous linear map with a finite-dimensional range is compact. While the converse is not true in general, we can say something to the effect that the range of a compact opeator is “almost finite-dimensional”. More precisely, the range of compact opeators can be approximated by a finite-dimensional subspace within a prescribed ε-distance, as described in the following result. The proof given below follows that of [8] closely.

Theorem 4.1. *Let $T : X \to Y$ be a compact linear transformation between Banach spaces X and Y. Then, given any $\varepsilon > 0$, there exists a finite-dimensional subspace M in $\mathsf{Ran}(T)$ such that, for any $x \in X$,*

$$\inf_{m \in M} \|Tx - m\| \le \varepsilon \|x\| .$$

Proof. Let $\varepsilon > 0$ be fixed but arbitrary. Let B_X denote the closed unit ball of X. Note that $T(B_X)$ is precompact, and thus can be covered with a finite cover, $\cup_1^N B(y_i, \varepsilon)$ with $y_i \in T(B_X) \subseteq \mathsf{Ran}(T)$. Now let M be the span of $y_1, \dots, y_N$, and note that $M \subseteq \mathsf{Ran}(T)$ and $dist(Tz, M) \le \varepsilon$ for any $z \in B_X$. Therefore, for any $x \in X$,

$$\inf_{m \in M} \left\| T\left(\frac{x}{\|x\|}\right) - m \right\| \le \varepsilon.$$

And thus,

$$\inf_{m' \in M} \|T(x) - m'\| \le \varepsilon \|x\| , \qquad m' = m \|x\| , m \in M.$$

□

5 Approximation by finite-dimensional operators

We have already noted that finite-dimensional operators on normed linear spaces are compact. Moreover, we know by Theorem 2.6 that the limit (in the operator norm) of a sequence of finite-dimensional operators is a compact operator. Moreover, we have seen that the range of compact operators can be approximated by a finite-dimensional subspace, in the sense described in Theorem 4.1. A natural follow up is the following question: Let $T : X \to Y$ be a compact operator between normed linear spaces X and Y, is it then true that T is a limit (in operator norm) of a sequence of finite-dimensional operators? The answer to this question, is negative for Banach spaces in general (see [3]). However, the result holds in the case Y is a Hilbert space, as given by the following known result:

Theorem 5.1. *Let $T : X \to Y$ be a compact operator, where X is a Banach space, and Y is a Hilbert space. Then, T is the limit (in operator norm) of a sequence of finite-dimensional operators.*

Proof. Let B_X denote the closed unit ball of X. Since T is compact, we know $T(B_X)$ is precompact; thus for any $n \ge 1$ there exists $y_1, \dots, y_N$ in $T(B_X)$ such that $T(B_X) \subseteq \cup_1^N B(y_i, 1/n)$. Let $M_n = span\{y_1, \dots, y_N\}$ and let Π_n be the orthogonal projection of Y onto M_n. First note that for any $y \in Y$, we have

$$\|\Pi_n(y) - y_i\| \le \|y - y_i\| , \quad i = 1, \dots, N. \tag{5.1}$$

Next define the mapping T_n by

$$T_n = \Pi_n \circ T.$$

We know, by construction, for any $x \in B_X$, $\|Tx - y_i\| \le 1/n$ for some $i \in \{1, \dots, N\}$. Moreover, by (5.1),

$$\|T_n x - y_i\| = \|\Pi_n(Tx) - y_i\| \le \|Tx - y_i\| \le 1/n.$$

Therefore, for any $x \in X$, with $\|x\| \leq 1$,

$$\|(T - T_n)x\| = \|Tx - T_n x\| \leq \|Tx - y_i\| + \|y_i - T_n x\| \leq 1/n + 1/n = 2/n,$$

and thus, $\|T - T_n\| \leq 2/n \to 0$ as $n \to \infty$. □

6 Spectrum of linear operators on a Banach space

Recall that for a linear operator A on a finite dimensional linear space, we define its spectrum $\sigma(A)$ as the set of its eigenvalues. On the other hand, for a linear operator T on an infinite dimensional (complex) normed linear space the spectrum $\sigma(T)$ of T is defined by,

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible in } B[X]\},$$

and $\sigma(T)$ is the disjoint union of the point spectrum $\sigma_p(T)$, (set of eigenvalues), contiuous spectrum, $\sigma_c(T)$, and residual spectrum, $\sigma_r(T)$. let us recall that the continous spectrum is given by,

$$\sigma_c(T) = \{\lambda \in \mathbb{C} : \mathsf{Ker}(T - \lambda I) = \{0\}, \, \mathsf{Ran}(T - \lambda I) \neq X, \, \overline{\mathsf{Ran}(T - \lambda I)} = X\},$$

and residual spectrum of T is given by,

$$\sigma_r(T) = \{\lambda \in \mathbb{C} : \mathsf{Ker}(T - \lambda I) = \{0\}, \, \overline{\mathsf{Ran}(T - \lambda I)} \neq X\}.$$

7 Some spectral properties of compact operators on Hilbert spaces

As we saw in Remark 2.8, a compact operator T on an infinite dimensional normed linear space X cannot be invertible in $B[X]$; therefore, we always have $0 \in \sigma(T)$. However, in general, not much can be said on whether $\lambda = 0$ is in point spectrum (i.e. an eigenvalue) or the other parts of the spectrum. However, we mention, the following special case:

Lemma 7.1. *Let H be a complex Hilbert space, and let $T \in B[H]$ be a one-to-one compact self-adjoint operator. Then, $0 \in \sigma_c(T)$.*

Proof. Since zero is not in the point spectrum, it must be in $\sigma_c(T)$ or in $\sigma_r(T)$. We show $0 \in \sigma_c(T)$ by showing that the range of T is dense in H. We first claim that $\mathsf{Ran}(T)^\perp = \{0\}$. To show this we proceed as follows. Let $z \in \mathsf{Ran}(T)^\perp$, and note that for every $x \in H$,

$$0 = \langle Tx, z \rangle = \langle x, Tz \rangle .$$

Hence, $Tz = 0$ which, since T is one-to-one, implies that $z = 0$. This shows that $\mathsf{Ran}(T)^\perp = \{0\}$. Thus, we have,

$$\overline{\mathsf{Ran}(T)} = \left(\mathsf{Ran}(T)^\perp\right)^\perp = \{0\}^\perp = H.$$ □

The next result sheds further light on the spectrum of a compact operator on a complex Hilbert space.

Lemma 7.2. *Let T be a compact operator on a complex Hilbert space H. Suppose λ is a non-zero complex number. Then,*

1. $\mathsf{Ker}(T - \lambda I)$ *is finite dimensional.*
2. $\mathsf{Ran}(T - \lambda I)$ *is closed.*
3. $T - \lambda I$ *is invertible if and only if* $\mathsf{Ran}(T - \lambda I) = H$.

Proof. The proof of this result is standard. See e.g. [8] for a proof. Here we just give the proof for the first statement of the theorem. Let $M = \mathsf{Ker}(T - \lambda I)$. Note that since T is continuous M is closed. Also, note that $T\mid_M = \lambda I$. We show M is finite-dimensional by showing that any bounded sequence in M has a convergent subsequence. Take a bounded sequence $\{x_n\}$ in M. Then, there exists a subsequence $\{Tx_{n_k}\}$ that converges. However, $Tx_{n_k} = \lambda x_{n_k}$, thus (also using $\lambda \neq 0$), it follows that $\{x_n\}$ has a convergent subsequence. □

Note that by the above theorem it follows immediately that a compact operator on a complex Hilbert space has empty non-zero continuous spectrum. In the next section, we will refine this result further by showing that in fact the same holds for the residual spectrum $\sigma_r(T)$ of a compact operator T on a complex Hilbert space. That is, if a non-zero $\lambda \in \mathbb{C}$ is in $\sigma(T)$ then it must be an eigenvalue of T.

8 Fredholm's Theorem of Alternative

The following well known result, known has Fredholm's theorem of alternative, has great utility in applications to integral equations:

Theorem 8.1. *Let H be a complex Hilbert space, and let $T \in B[H]$ be compact and let λ be a nonzero complex number. Then, exactly one of the followings hold:*

1. *$T - \lambda I$ is invertible*
2. *λ is an eigenvalue of T.*

Proof. Assume that the first statement in not true. We show that $\lambda \in \sigma_p(T)$. We have already seen that λ cannot be in continuous spectrum of T. Suppose now that $\lambda \in \sigma_r(T)$. Then, $\mathsf{Ran}(T - \lambda I) \neq H$. Thus, there is a non-zero $x \in \mathsf{Ran}(T - \lambda I)^\perp = \mathsf{Ker}(T^* - \bar{\lambda} I)$. That is, $\bar{\lambda} \in \sigma_p(T^*)$. Now, by Lemma 7.2, $\mathsf{Ran}(T^* - \bar{\lambda} I)$ is closed and since $T^* - \bar{\lambda} I$ is not invertible, by the same Lemma, $\mathsf{Ran}(T^* - \bar{\lambda} I) \neq H$. Therefore,

$$\mathsf{Ker}(T - \lambda I) = \mathsf{Ran}(T^* - \bar{\lambda} I)^\perp \neq \{0\}.$$

That is, $\lambda \in \sigma_p(T)$, contradicting the supposition that $\lambda \in \sigma_r(T)$. Therefore, it follows that $\lambda \in \sigma_p(T)$. □

Let us note the following interpretation of the above result. Let T and $\lambda \neq 0$ be as in the above theorem. Then, either $T - \lambda I$ is invertible or λ is an eigenvalue of T. Also, using Lemma 7.2, we know that the eigenspace corresponding to λ is finite-dimensional. That is $T - \lambda I$ is invertible, except possibly on a finite-dimensional subspace of H. Therefore, and equation of form,

$$(T - \lambda I)x = y,$$

is solvable uniquely, modulo a finite-dimensional subspace of H.

9 Spectral theorem for compact self-adjoint operators

Compact self-adjoint operators on infinite dimensioal Hilbert spaces resemble many properties of the symmetric matrices. Of particular interest is the spectral decomposition of a compact self-adjoint operator as given by the following (see e.g., [8] for more details):

Theorem 9.1. *Let H be a (real or complex) Hilbert space and let $T : H \to H$ be a compact self-adjoint operator. Then, H has an orthonormal basis $\{e_i\}$ of eigenvectors of T corresponding to eigenvalues λ_i. In addition, the following holds:*

1. *The eigenvalues λ_i are real having zero as the only possible point of accumulation.*
2. *The eigenspaces corresponding to distinct eigenvalues are mutually orthogonal.*
3. *The eigenspaces corresponding to non-zero eigenvalues are finite-dimensional.*

In the case of a positive compact self-adjoint operator, we know that the eigenvalues are non-negative. Hence, we may order the eigenvalues as follows

$$\lambda_1 \geq \lambda_2 \geq ... \geq 0.$$

Using the spectral theorem, we can "diagonalize" a compact self-adjoint operator $T : H \to H$, as follows,

$$Tu = \sum_{j=1}^{\infty} \lambda_j \langle u, e_j \rangle e_j, \quad u \in H,$$

where we also can show that (using compactness of T) that $\lambda_j \to 0$ as $j \to \infty$.

Notice that for a strictly positive compact selfadjoint operator we can write the inverse operator by

$$T^{-1}u = \sum_{j=1}^{\infty} \lambda_j^{-1} \langle u, e_j \rangle e_j.$$

This is a densely defined unbounded operator. It is densely defined because as we saw before range of T is dense in H. The unboundedness follows from the earlier result that states compact operators do not have a bounded inverse. We can also see the unboundedness of this operator directly as follows:

$$\|Te_i\| = \left\| \sum_{j=1}^{\infty} \lambda_j^{-1} \langle e_i, e_j \rangle e_j \right\| = \left\| \lambda_i^{-1} e_i \right\| = \lambda_i^{-1} \to \infty,$$

as $i \to \infty$.

10 Singular value decomposition and some basic results

Here we briefly recall the notion of the singular value decomposition (SVD) of compact linear transformations between Hilbert spaces.

First we recall the following definition [6]:
Definition 10.1. *Let H and K be Hilbert spaces, and let $A : H \to K$ be a compact linear transformation, and $A^* : K \to H$ be its adjoint. The nonnegative square roots of the eigenvalues of the (positive self-adjoint) compact operator $A^*A : H \to H$ are called the singular values of A.*

The following result describes the singular value decomposition of compact linear transformations [6].
Theorem 10.2. *Let H and K be real Hilbert spaces, and let $A : H \to K$ be a (nonzero) compact operator. Let $\{\sigma_n\}$ be the sequence of nonzero singular values of A repeated according to their multiplicities and in descending order. Then, there exist orthonormal sequences $\{v_n\}$ and $\{u_n\}$ in H and K, respectively, such that*

$$Av_n = \sigma_n u_n, \quad A^*u_n = \sigma_n v_n, \quad n \geq 1.$$

For each $x \in H$, we have

$$x = \sum_{n=1}^{\infty} \langle x, v_n \rangle_H v_n + Qx, \tag{10.1}$$

where Q is the orthogonal projection operator onto the kernel of A. Moreover,

$$Ax = \sum_{n=1}^{\infty} \sigma_n \langle x, v_n \rangle_H u_n, \quad x \in H. \tag{10.2}$$

We can use the SVD of a compact operator $A : H \to K$ in a similar way as we do with matrices. Let us for example note, the following result regarding the operator norm of A. Recall that for $A \in B[H, K]$, the operator norm is given by

$$\|A\| = \sup_{\|x\|_H = 1} \|Ax\|_K .$$

Theorem 10.3. *Let $A : H \to K$ be compact, and let $\{\sigma_n\}$ be the sequence of its nonzero singular values ordered according to*

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots$$

Then, $\|A\| = \sigma_1$.

Proof. Let $x \in H$ be arbitrary and consider its decomposition (10.1). We note that

$$\|x\|_H^2 = \sum_n \langle x, v_n \rangle_H^2 + \|Qx\|_H^2$$

Moreover, we have $Ax = \sum_n \sigma_n \langle x, v_n \rangle_H u_n$, and thus

$$\|Ax\|_K^2 = \sum_n \sigma_n^2 \langle x, v_n \rangle_H^2 \leq \sigma_1^2 \sum_n \langle x, v_n \rangle_H^2 \leq \sigma_1^2 \|x\|_H^2 .$$

Thus, we have that $\|Ax\|_K \leq \sigma_1 \|x\|_H$, for every $x \in H$. Moreover, $\|Av_1\|_K = \sigma_1 \|u_1\|_K = \sigma_1$. Thus, $\|A\| = \sigma_1$. □

The following result is an analogue of best rank-k approximation result for matrices. The proof given below follows in similar lines as that given in [1].

Theorem 10.4. *Let $A : H \to K$ be compact. Consider SVD of A,*

$$Ax = \sum_{n=1}^{\infty} \sigma_n \langle x, v_n \rangle u_n, \quad x \in H,$$

as in Theorem 10.2. We have,

$$\sigma_k = \min\{\|A - F\| : \mathsf{rank}(F) \leq k - 1\}, \quad k = 1, 2, \ldots .$$

Proof. The result is immediate for $k = 1$. Let $k > 1$, and let $F \in B[H, K]$ have rank at most $k - 1$, and consider $\{v_1, \ldots, v_k\}$. Since $\mathsf{rank}(F) \leq k - 1$, the set $\{Fv_1, \ldots, Fv_k\}$ must be linearly dependent. That is there exists, $\alpha_1, \ldots, \alpha_k$, not all zeros, with $0 = \sum_{n=1}^{k} \alpha_n F v_n = F(\sum_{n=1}^{k} \alpha_n v_n)$. Let $z = \sum_{n=1}^{k} \alpha_n v_n$, and let $x = z / \|z\|_H$, so that $\|x\|_H = 1$ and $Fx = 0$. Now,

$$\|A - F\|^2 \geq \|(A - F)x\|_K^2 = \|Ax\|_K^2 = \sum_{n=1}^{k} \sigma_n^2 \langle x, v_n \rangle_H^2 \geq \sigma_k^2 \|x\|_H^2 = \sigma_k^2.$$

Therefore, $\|A-F\| \geq \sigma_k$. Next, define

$$F = \sum_{n=1}^{k-1} \sigma_n \langle \cdot, v_n \rangle u_n.$$

Clearly, F has rank $k-1$, and we have $A - F = \sum_{n=k}^{\infty} \sigma_n \langle \cdot, v_n \rangle u_n$, from which we have $\|A-F\| = \sigma_k$. □

The following singular value inequality is useful in applications, and is analogue of results for matrices. The proof here is adapted from that in [1, page 188].

Theorem 10.5. *Let $A : H \to K$ be compact and assume $B \in B[H]$ Then,*

$$\sigma_k(AB) \leq \sigma_k(A) \|B\|, \quad k \geq 1.$$

Proof. First note that by the assumptions of the theorem AB is compact in $B[H,K]$. Consider,

$$A = \sum_{n=1}^{\infty} \sigma_n \langle \cdot, v_n \rangle u_n, \quad F = \sum_{n=1}^{k-1} \sigma_n \langle \cdot, v_n \rangle u_n.$$

Note that $\mathsf{rank}(FB) \leq k-1$; therefore, $\sigma_k(AB) \leq \|AB - FB\| \leq \|B\| \|A-F\| = \sigma_k(A) \|B\|$. □

11 A numerical illustration

Here we study the convolution operator, $F : L^2([0,1]) \to L^2([0,1])$ given by

$$(Fu)(x) = \int_0^1 k(x-y)u(y)dy, \quad k(x) = \frac{1}{\sqrt{2\pi}\gamma} \exp(-\frac{x^2}{2\gamma^2}), \quad \gamma = 0.03.$$

Notice that this is a compact operator (cf. e.g., Lemma 3.1). To study the spectral properties of F, we compute the eigenvalues and eigenvectors of the operator numerically by first discretizing the integral operator via quadrature and then computing the spectrum of the discretized operator. Using an n-point composite mid-point rule we obtain the discretized operator $\mathbf{K}$:

$$K_{ij} = hC \exp\left(\frac{-\big((i-j)h\big)^2}{2\gamma^2}\right), \quad h = 1/n,$$

with $i, j \in \{1, \ldots, n\}$. In the results reported here, we used $n = 128$. Figure 1 shows the eigenvalues of $\mathbf{K}$, and Figure 2 shows the eigenvectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, and e_{10}. Notice that the higher order eigenvectors (eigenfunctions) become more oscillatory.

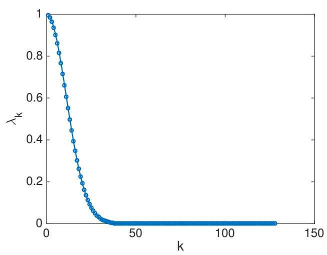

Figure 1: Eigenvalues of the convolution operator F.

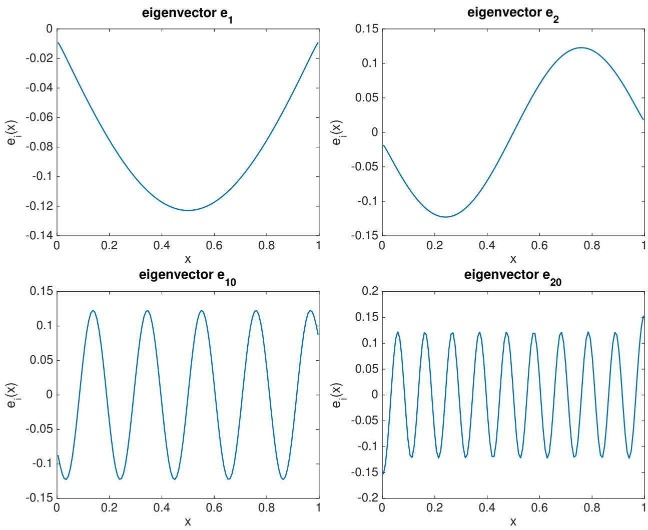

Figure 2: A few of the eigenvectors of the convolution operator F. We plot eigenvectors corresponding to the largest eigenvalue λ_1, second largest eigenvalue λ_2, as well as those corresponding to λ_{10} and λ_{20}.

Applying the compact operator F to a function in $L^2([0,1])$ has a smoothing effect. This is illustrated in Figure 3, where we show the effect of applying F on two different source functions.

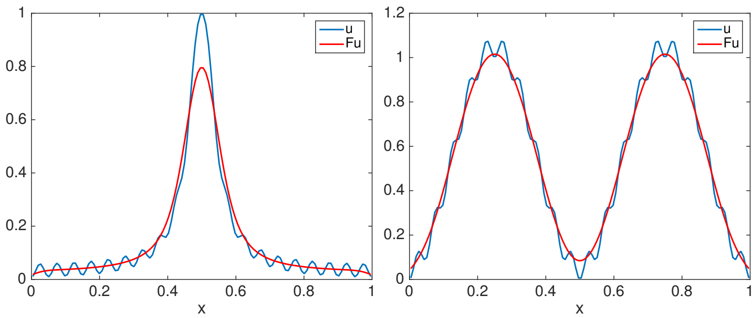

Figure 3: We illustrate the smoothing effect of the convolution operator by applying it to two different source function.

References

[1] Rajendra Bhatia. *Notes on functional analysis*, volume 50 of *Texts and Readings in Mathematics*. Hindustan Book Agency, New Delhi, 2009.

[2] John B Conway. *A course in operator theory*. American Mathematical Soc., 2000.

[3] Per Enflo. A counterexample to the approximation problem in Banach spaces. *Acta Math.*, 130:309–317, 1973.

[4] Avner Friedman. *Foundations of modern analysis*. Courier Corporation, 1970.

[5] Israel Gohberg, Seymour Goldberg, and Marinus Kaashoek. *Basic classes of linear operators*. Birkhäuser, 2012.

[6] Rainer Kress. *Linear integral equations*, volume 82. Springer, 2014.

[7] Erwin Kreyszig. *Introductory functional analysis with applications*, volume 81. wiley New York, 1989.

[8] Arch W. Naylor and George Sell. *Linear operator theory in engineering and science*. Springer-Verlag, New York, 1982.

[9] Michael Reed and Barry Simon. *Functional Analysis, volume 1 of Methods of Modern Mathematical Physics*. Academic Press, New York, 1972.