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Irreducibility of a symmetry group implies isotropy

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Abstract We derive minimal conditions on a symmetry group of a linearly elastic material that implies its isotropy. A natural setting for the formulation and analysis is provided by the group representation theory where the necessary and sufficient conditions for isotropy are expressed in terms of the irreducibility of certain group representations. We illustrate the abstract results by (re)deriving several old and new theorems within a unified theory.

Keywords linear elasticity · isotropy · symmetry groups · group representation · Schur's Lemma · random media

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1 Introduction

In the linear theory of elasticity, the elasticity tensor C (at a material point $\mathbf{x}$) is a linear map (self-adjoint in the case of hyperelasticity) on the space *Sym* of symmetric tensors. The strain E and stress S at $\mathbf{x}$ are related through $S = \mathsf{C}[E]$. The material at $\mathbf{x}$ is isotropic if

$$Q\mathsf{C}[E]Q^T = \mathsf{C}[QEQ^T] \quad \text{for all } E \in \mathit{Sym},\ Q \in \mathit{Orth}, \tag{1}$$

where *Orth* is the set of all orthogonal linear transformations of the space. The identity in (1) implies that C necessarily is of the form

$$\mathsf{C}[E] = 2\mu E + \lambda(\operatorname{tr} E)I, \tag{2}$$

where the scalars μ and λ are the *Lamé moduli* at $\mathbf{x}$. This is a classic result, going back to Cauchy and Navier in 1823. See the footnote to equation (1.1) in [26] (or its reprint [27]) for a brief historical survey. The "canonical" modern proof is a corollary to the Ericksen-Rivlin representation theorem for general (not necessarily linear) isotropic tensors; see [29,

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page 33] and [9, page 196]. Nonetheless, novel and insightful proofs continue to appear to this day. The articles [8, 19, 3, 6, 2, 7, 17, 23, 30, 4, 13, 18, 16] (in the chronological order) provide a (non-exhaustive) sampling of this and related results that have appeared in this journal between 1974 and 2006.

Our objective in this article is to see to what extent can the condition "for all $Q \in Orth$" in (1) be relaxed while preserving the conclusion of isotropy. Specifically, let $G \subseteq Orth$ be a *symmetry group* for the material in the sense that,

$$Q\mathsf{C}[E]Q^T = \mathsf{C}[QEQ^T] \quad \text{for all } E \in Sym,\ Q \in G. \tag{3}$$

We ask: *Under what minimal conditions on G does* (3) *imply* (2)*?* We show that group representation theory provides the natural framework to answer this question. Theorem 1 of Section 7 establishes a connection between Schur's Lemma in group representation theory and symmetry groups in elasticity and shows that irreducibility of certain representations of G implies the isotropy of C in n-dimensional hyperelasticity. Theorem 2 provides a converse. Theorems 3 and 4 extend Theorem 1 to the non-hyperelastic and polar media, respectively, in three dimensions.

We provide several illustrations of the applications of these theorems. In Section 8, we explain the connection between Theorem 1 and the classical representation for isotropic elasticity tensors as in equation (2). This essentially amounts to verifying irreducibility of certain representations of *Orth*, which is implied by a result in [19]. In Theorem 7 of Section 9 we show that in plane hyperelasticity, the symmetry group generated by a single generic rotation implies isotropy. Here "generic" means a rotation other than an integer multiple of $\pi/2$. In the non-hyperelastic case one needs a symmetry group generated by a generic rotation and a reflection; see Theorem 8. In Theorem 10 of Section 10 we reproduce, in the context of group representations, a result obtained in [11], whereby symmetry under a group generated by a pair of generic rotations in three dimensions implies isotropy. The precise meaning of "generic" is given in Definition 1 of Section 10. In Theorem 11 of Section 11, we employ a variant of Schur's Lemma to generalize Theorem 10 to fourth order tensors C defined on the space of *all* second order tensors (not necessarily symmetric). This is related to questions studied in [6], [2], and [23].

We have strived to present the arguments clearly, and avoid undue generality lest it obfuscate the underlying ideas. For this reason, we have restricted Theorem 1 to the hyperelastic case but we present remarks and extensions, where instructive, to the non-hyperelastic case throughout the article. In this connection, it is worth nothing, cf. Truesdell [28, Lecture 27], that in isothermal thermoelasticity, the second law of thermodynamics implies hyperelasticity. Carroll [1] brings this out by constructing a rather simple non-hyperelastic material that generates net excess energy in a cyclic process.

2 Motivation for this analysis

In the elasticity literature, it is common to limit the symmetry group G in (3) to the crystallographic groups. There *are*, however, interesting situations where non-traditional symmetries arise. One such instance is in the case of the homogenized limit of media with random microstructures. For extensive surveys of basic principles and contemporary research on such media see [21] and [25].

Consider, for instance, a two-dimensional random medium that fills the entire plane. The material consists of of a *matrix* with embedded *granules*. Both the matrix and granules are homogeneous and isotropic—think of glass beads embedded in a rubber sheet. The granules are identical regular pentagons oriented such that their edges are pairwise parallel. The centers of the pentagons are distributed randomly in the plane according to a Poisson distribution, that is, the probability of having k pentagon centers in a region is $a^k e^{-a}/k!$, where a is proportional to the area of the region. Such a material has a five-fold symmetry in the probabilistic sense: any realization and its $2\pi/5$ radian rotation are equally likely. Let Ω be the probability space corresponding to all such realizations. Thus, a point ω in Ω represents a particular realization. Let $\mathsf{C}_{\mathbf{x},\omega}$ be that realization's elasticity tensor at the point $\mathbf{x}$ in space. For any $\varepsilon > 0$ define a scaled material with the elasticity tensor $\mathsf{C}^{(\varepsilon)}_{\mathbf{x},\omega} = \mathsf{C}_{\mathbf{x}/\varepsilon,\omega}$. As $\varepsilon \to 0$, we obtain a family of elastic materials with progressively finer microstructure. It is possible to show (see [22, 15]) that under reasonable assumptions on the nature of the probabilistic distribution of $\mathsf{C}_{\mathbf{x}/\varepsilon,\omega}$, the solution u_ε of the boundary value problem of elasticity on any bounded domain $\mathscr{B}$:

$$\begin{aligned} \operatorname{div}\left(\mathsf{C}^{(\varepsilon)}_{\mathbf{x},\omega}\left[\nabla u_\varepsilon(\mathbf{x},\omega)\right]\right) + f(\mathbf{x}) = 0 \quad &\text{on } \mathscr{B}, \\ u_\varepsilon = 0 \quad &\text{on } \partial\mathscr{B} \end{aligned}$$

converges (in a suitable sense), as $\varepsilon \to 0$, to the solution u_0 of the limiting problem:

$$\begin{aligned} \operatorname{div}\left(\mathsf{C}^{(0)}\left[\nabla u_0(\mathbf{x})\right]\right) + f(\mathbf{x}) = 0 \quad &\text{on } \mathscr{B}, \\ u_0 = 0 \quad &\text{on } \partial\mathscr{B}, \end{aligned}$$

where $\mathsf{C}^{(0)}$ is a *constant* elasticity tensor, called the *homogenized limit* of the family $\mathsf{C}^{(\varepsilon)}_{\mathbf{x},\omega}$.

It is possible to show that (see [15]) the homogenized limit, $\mathsf{C}^{(0)}$, inherits the symmetry properties of $\mathsf{C}_{\mathbf{x},\omega}$. Thus the five-fold symmetry of $\mathsf{C}_{\mathbf{x},\omega}$ implies that

$$Q\mathsf{C}^{(0)}[E]Q^T = \mathsf{C}^{(0)}[QEQ^T] \quad \text{for all } E \in \mathit{Sym},\ Q \in G_5 \tag{4}$$

where G_5 is the group corresponding to $2\pi/5$ radian rotations in the plane. Note that this symmetry group is not a crystallographic class. It is a consequence of Theorem 7 of Section 9 that the identity (4) implies that $\mathsf{C}^{(0)}$ is isotropic.

3 Notation

Throughout this article we use the following notations and conventions. We write $\mathscr{X}$ for a generic finite-dimensional vector space equipped with an inner product $\langle\cdot,\cdot\rangle$. The corresponding scalar field may be real or complex; we will be explicit when it does matter. We write $\mathit{Lin}(\mathscr{X})$ for the space of linear operators on $\mathscr{X}$. The *general linear group* $\mathit{GL}(\mathscr{X})$ of $\mathscr{X}$ is the group of all invertible operators in $\mathit{Lin}(\mathscr{X})$.

We let I denote the identity operator in $\mathit{Lin}(\mathscr{X})$, thus $I\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathscr{X}$, and we set

$$\mathit{Sph}(\mathscr{X}) = \operatorname{span}\{I\} = \left\{L \in \mathit{Lin}(\mathscr{X}) : L = \alpha I \text{ for a scalar } \alpha\right\}.$$

To any $A \in \mathit{Lin}(\mathscr{X})$ there corresponds a unique *adjoint* $A^* \in \mathit{Lin}(\mathscr{X})$ such that $\langle A\mathbf{x},\mathbf{y}\rangle = \langle\mathbf{x},A^*\mathbf{y}\rangle$ for all $\mathbf{x},\mathbf{y} \in \mathscr{X}$. A map A is *self-adjoint* if $A^* = A$. A map A is *unitary* if $AA^* = I$. If $\mathscr{X}$ is a real vector space, the adjoint of A is denoted by A^T, and unitary maps are called *orthogonal* maps. Let $\mathit{Orth}(\mathscr{X})$ be the set of all orthogonal maps on $\mathscr{X}$.

We write $Sym(\mathscr{X})$ for the subspace of self-adjoint maps in $Lin(\mathscr{X})$. The inner product $\langle\cdot,\cdot\rangle$ of $\mathscr{X}$ induces a natural inner product on *Lin* defined by $\langle\langle A,B\rangle\rangle = \mathrm{tr}(A^*B) = \mathrm{tr}(AB^*)$, where tr is the trace operator.[1] Let us note that $\mathrm{tr}A = \langle\langle I,A\rangle\rangle$ for every $A \in Lin(\mathscr{X})$.

We let $Dev(\mathscr{X})$ be the orthogonal complement of $Sph(\mathscr{X})$ in $Sym(\mathscr{X})$, that is,

$$Sym(\mathscr{X}) = Sph(\mathscr{X}) \oplus Dev(\mathscr{X}). \tag{5}$$

If $A \in Sym(\mathscr{X})$, it follows from $\mathrm{tr}A = \langle\langle I,A\rangle\rangle$ that $A \in Dev(\mathscr{X})$ if and only if $\mathrm{tr}A = 0$. That is, the space $Dev(\mathscr{X})$ consists of the subspace of traceless operators in $Sym(\mathscr{X})$. The decomposition in (5) for an element A of $Sym(\mathscr{X})$ takes the form

$$A = \frac{1}{n}(\mathrm{tr}A)I + \big(A - \frac{1}{n}(\mathrm{tr}A)I\big),$$

where n is the dimension of the vector space $\mathscr{X}$. This is akin to the decomposition of a stress into a "hydrostatic pressure" plus a "deviatoric stress" and should explain the notation *Dev*, which was introduced in [19], if not earlier.

We denote by $Skew(\mathscr{X})$ the subspace of skew-symmetric transformations in $Lin(\mathscr{X})$, that is, $Skew(\mathscr{X}) = \{A \in Lin(\mathscr{X}) : A^* = -A\}$. Let us note that $Lin(\mathscr{X})$ has the orthogonal decomposition

$$Lin(\mathscr{X}) = Sym(\mathscr{X}) \oplus Skew(\mathscr{X}) = Sph(\mathscr{X}) \oplus Dev(\mathscr{X}) \oplus Skew(\mathscr{X}).$$

The *tensor product* of elements $\mathbf{u}$ and $\mathbf{v}$ in $\mathscr{X}$, denoted by $\mathbf{u}\otimes\mathbf{v}$, is a linear mapping on $\mathscr{X}$ defined by

$$(\mathbf{u}\otimes\mathbf{v})\mathbf{x} = \langle\mathbf{x},\mathbf{v}\rangle\mathbf{u}, \quad \forall\mathbf{x}\in\mathscr{X}.$$

In the rest of this article, the abstract space $\mathscr{X}$ will represent concrete vector spaces of various types. In the special case when $\mathscr{X} = \mathscr{V}$, where $\mathscr{V}$ is the usual Euclidean vector space, we simplify the notations by writing *Lin*, *Sym*, *Dev*, *Skew*, and *Orth* instead of $Lin(\mathscr{V})$, $Sym(\mathscr{V})$, $Dev(\mathscr{V})$, $Skew(\mathscr{V})$, and $Orth(\mathscr{V})$.

4 Complexification

Complexification provides a procedure to extend a real vector space to a complex one. For the reader's convenience, we briefly describe the process. See [10] for a full treatment.

4.1 Complexification of a vector space

The complexification $\mathscr{X}^\dagger$ of a real vector space $\mathscr{X}$ is the set of all ordered pairs $(\mathbf{u},\mathbf{v})$ with $\mathbf{u}$ and $\mathbf{v}$ in $\mathscr{X}$. Addition and scalar multiplication (by complex numbers) on $\mathscr{X}^\dagger$ are defined as follows. For all $\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{z}$ in $\mathscr{X}$ and a, b in $\mathbf{R}$:

$$(\mathbf{u},\mathbf{v}) + (\mathbf{w},\mathbf{z}) = (\mathbf{u}+\mathbf{w},\mathbf{v}+\mathbf{z}),$$
$$(a+bi)(\mathbf{u},\mathbf{v}) = (a\mathbf{u}-b\mathbf{v}, b\mathbf{u}+a\mathbf{v}).$$

[1] The trace of a linear operator on a finite dimensional vector space is the sum of the diagonal terms in its matrix representation with respect to an orthonormal basis. The sum is independent of the choice of basis, and therefore the trace is an intrinsic property of the operator. The trace also equals the sum of the operator's eigenvalues (including the complex ones); see [10] for details.

Note that $i(\mathbf{u},0)=(0,\mathbf{u})$ for every $\mathbf{u}\in\mathscr{X}$, and in general, every element $(\mathbf{u},\mathbf{v})\in\mathscr{X}^\dagger$ can be written as

$$(\mathbf{u},\mathbf{v})=(\mathbf{u},0)+i(\mathbf{v},0). \tag{6}$$

If $\{\mathbf{e}_i\}_1^n$ is a basis of $\mathscr{X}$, then $\{(\mathbf{e}_i,0)\}_1^n$ is a basis of $\mathscr{X}^\dagger$, therefore $\mathscr{X}$ and $\mathscr{X}^\dagger$ have the same dimension.

The linear mapping $\mathscr{X}\to\mathscr{X}^\dagger:\mathbf{u}\mapsto(\mathbf{u},0)$ (over the scalar field $\mathbf{R}$) is the natural embedding of $\mathscr{X}$ into $\mathscr{X}^\dagger$. It is often convenient to identify an element $\mathbf{u}\in\mathscr{X}$ with its copy $(\mathbf{u},0)\in\mathscr{X}^\dagger$. Thus, in view of this identification and (6) we write $\mathbf{u}$ for $(\mathbf{u},0)$ and $\mathbf{u}+i\mathbf{v}$ for $(\mathbf{u},\mathbf{v})\in\mathscr{X}^\dagger$.

If $\mathscr{X}$ is endowed with an inner product $\langle\cdot,\cdot\rangle$, we get a corresponding inner product on $\mathscr{X}^\dagger$ given by

$$\langle\mathbf{u}+i\mathbf{v},\mathbf{w}+i\mathbf{z}\rangle=\big[\langle\mathbf{u},\mathbf{w}\rangle+\langle\mathbf{v},\mathbf{z}\rangle\big]+i\big[\langle\mathbf{v},\mathbf{w}\rangle-\langle\mathbf{u},\mathbf{z}\rangle\big].$$

4.2 Complexification of linear transformations

Let $\mathscr{X}$ be a real vector space as before. Any linear mapping $T:\mathscr{X}\to\mathscr{X}$ can be extended to a linear mapping $T^\dagger:\mathscr{X}^\dagger\to\mathscr{X}^\dagger$ according to:

$$T^\dagger(\mathbf{u}+i\mathbf{v})=T\mathbf{u}+iT\mathbf{v},\quad\forall\mathbf{u},\mathbf{v}\in\mathscr{X}.$$

The complexified version of a linear transformation behaves essentially the same way as its real counterpart. Here we note a few facts which will be relevant to our subsequent analysis. Let T be a linear mapping on $\mathscr{X}$. If $T=\alpha I$, for some $\alpha\in\mathbf{R}$, then clearly $T^\dagger=\alpha I$ also. Conversely, if $T^\dagger=\sigma I$, then $\sigma\in\mathbf{R}$ and $T=\sigma I$. Also, for linear mappings T and S, we have $TS=ST$ if and only if $T^\dagger S^\dagger=S^\dagger T^\dagger$.

Eigenspaces of T and $T^\dagger$ are closely related. If α is a real eigenvalue of $T^\dagger$, then it is also an eigenvalue of T. Also, if $(a\pm ib,\mathbf{x}\pm i\mathbf{y})$ are complex eigenpairs for $T^\dagger$, it is simple to see that $\operatorname{span}\{\mathbf{x},\mathbf{y}\}$ is an invariant subspace of T.

When it is clear from the context, we drop the superscript † for the complexification of a linear transformation. For example, if Q is an orthogonal transformation on $\mathbf{R}^3$, we refer to its complexification (a unitary map on $\mathbf{C}^3$) also as Q.

5 Group representation

Let $\mathscr{X}$ be a vector space and G be a group with the group operation $*$. A *group homomorphism* from G to $GL(\mathscr{X})$ is called a *representation* of G on $\mathscr{X}$. Thus $\Pi:G\to GL(\mathscr{X})$ is a representation if

$$\Pi(g_1*g_2)=\Pi(g_1)\circ\Pi(g_2)\quad\text{for all } g_1,g_2\in G,$$

where the $\circ$ denotes the composition of operators on *Lin*. A representation Π of a group G on an inner product space $(\mathscr{X},\langle\cdot,\cdot\rangle)$ is said to be *orthogonal* if $\Pi(g)\in Orth(\mathscr{X})$ for every $g\in G$; that is, Π is orthogonal if

$$\langle\Pi(g)\mathbf{u},\Pi(g)\mathbf{v}\rangle=\langle\mathbf{u},\mathbf{v}\rangle\quad\text{for all }\mathbf{u},\mathbf{v}\in\mathscr{X},\ g\in G. \tag{7}$$

For an accessible introduction to group representation theory see [12].

5.1 Invariance and irreducibility

Let Π be a representation of a group G on a vector space $\mathscr{X}$. A subspace U of $\mathscr{X}$ is said to be *invariant under* Π if

$$\Pi(g)U \subseteq U \quad \forall g \in G.$$

A representation is *irreducible* if its only invariant subspaces are $\{0\}$ and $\mathscr{X}$. In other words, the representation Π is irreducible if for any subspace U of $\mathscr{X}$ we have:

$$\Big\{\Pi(g)U \subseteq U \quad \forall g \in G\Big\} \quad \Rightarrow \quad \Big\{U = \{0\} \text{ or } U = \mathscr{X}\Big\}.$$

Remark 1 In finite-dimensional spaces, the inclusion $\Pi(g)U \subseteq U$ may be replaced by equality $\Pi(g)U = U$ without affecting the definition.

Irreducible representations are important as they can be thought of building blocks for more complicated representations.

5.2 Representations of orthogonal groups

Here we introduce a few special representations of subgroups of the orthogonal group that play significant roles in our work. All of these representations are *orthogonal* in the sense of the definition in equation (7). In what follows, $\mathscr{V}$ is the n-dimensional Euclidean vector space and G is a subgroup of *Orth*. Note that following the convention established in Section 3, we write *Orth* instead of $Orth(\mathscr{V})$, etc.

The *natural representation*, Π_{nat}, of G on $\mathscr{V}$ is the mapping $\Pi_{\text{nat}} : G \to GL(\mathscr{V})$ defined by

$$\Pi_{\text{nat}}(Q) = Q \quad \text{for all } Q \in G. \tag{8}$$

The *adjoint representation*, Π_{adj}, of G on *Lin* is the mapping $\Pi_{\text{adj}} : G \to GL(Lin)$ defined by

$$\Pi_{\text{adj}}(Q)[A] = QAQ^T \quad \text{for all } Q \in G,\ A \in Lin. \tag{9}$$

To see that Π_{adj} is a group homomorphism, we note that for any $A \in Lin$ and $Q_1, Q_2 \in G$ we have:

$$\begin{aligned}\Pi_{\text{adj}}(Q_1Q_2)[A] &= (Q_1Q_2)A(Q_1Q_2)^T = Q_1(Q_2AQ_2^T)Q_1^T \\ &= Q_1\big(\Pi_{\text{adj}}(Q_2)[A]\big)Q_1^T = \Pi_{\text{adj}}(Q_1)\big[\Pi_{\text{adj}}(Q_2)[A]\big] = \big(\Pi_{\text{adj}}(Q_1)\Pi_{\text{adj}}(Q_2)\big)[A].\end{aligned}$$

Thus $\Pi_{\text{adj}}(Q_1Q_2) = \Pi_{\text{adj}}(Q_1)\Pi_{\text{adj}}(Q_2)$, as required by the group homomorphism property. To show orthogonality, for every $Q \in G$ and $A, B \in Lin$ we compute:

$$\langle\langle \Pi_{\text{adj}}(Q)[A], \Pi_{\text{adj}}(Q)[B] \rangle\rangle = \langle\langle QAQ^T, QBQ^T \rangle\rangle = \text{tr}(QAQ^TQB^TQ^T) = \text{tr}(AB^T) = \langle\langle A, B \rangle\rangle.$$

Additionally, Π_{adj} has the following invariance properties:

1. For every $Q \in G$, $\Pi_{\text{adj}}(Q)$ leaves the subspaces *Sym* and *Skew* of *Lin* invariant.
2. For every $Q \in G$, $\Pi_{\text{adj}}(Q)$ leaves the subspaces *Sph* and *Dev* of *Sym* invariant. Invariance of *Sph* is trivial. To see that $\Pi_{\text{adj}}(Q)$ leaves *Dev* invariant, note that $\text{tr}\big(\Pi_{\text{adj}}(Q)[A]\big) = \text{tr}(QAQ^T) = \text{tr}A$. Therefore, $A \in Dev$ if and only if $\Pi_{\text{adj}}(Q)[A] \in Dev$.

These motivate the following additional definitions:

The *symmetric representation*, Π_{sym}, of G on *Sym* is the mapping $\Pi_{\text{sym}} : G \to GL(Sym)$ defined by

$$\Pi_{\text{sym}}(Q)[A] = QAQ^T \quad \text{for all } Q \in G,\ A \in Sym. \tag{10}$$

The *deviatoric representation*, Π_{dev}, of G on *Dev* is the mapping $\Pi_{\text{dev}} : G \to GL(Dev)$ defined by

$$\Pi_{\text{dev}}(Q)[A] = QAQ^T \quad \text{for all } Q \in G,\ A \in Dev. \tag{11}$$

The *skew representation*, Π_{skw}, of G on *Skew* is the mapping $\Pi_{\text{skw}} : G \to GL(Skew)$ defined by

$$\Pi_{\text{skw}}(Q)[W] = QWQ^T \quad \text{for all } Q \in G,\ W \in Skew. \tag{12}$$

Remark 2 The invariance relation (3) may be expressed in terms of the group representation Π_{sym} as:

$$\Pi_{\text{sym}}(Q) \circ \mathsf{C} = \mathsf{C} \circ \Pi_{\text{sym}}(Q), \quad \forall Q \in G. \tag{13}$$

Thus, the invariance relation (3) is equivalent to asserting that C and Π_{sym} commute. This commutativity condition appears explicitly in equation (5) of [19] and subsequent literature, albeit outside the group representation context.

Remark 3 If C is non-self-adjoint, then it follows from the identity in (13) that $\mathsf{C}^T \circ \Pi_{\text{sym}}(Q)^T = \Pi_{\text{sym}}(Q)^T \circ \mathsf{C}^T$. But since $\Pi_{\text{sym}}(Q)$ is orthogonal, this simplifies to

$$\Pi_{\text{sym}}(Q) \circ \mathsf{C}^T = \mathsf{C}^T \circ \Pi_{\text{sym}}(Q), \quad \forall Q \in G. \tag{14}$$

5.3 Complexification of a representation

Let Π be a representation of a group G on a real vector space $\mathscr{X}$. Its complexification $\Pi^\dagger$, defined by

$$\Pi^\dagger(g) = \big(\Pi(g)\big)^\dagger, \quad \forall g \in G,$$

is a representation of G on $\mathscr{X}^\dagger$. It is straightforward to verify that if $\Pi^\dagger$ is irreducible, then so is Π; however, the converse is not true. For instance, let Q be the rotation of $\mathbf{R}^2$ by an angle $\theta \in (0,\pi)$ and let G be the group generated[2] by Q. The natural representation Π_{nat} of G on $\mathbf{R}^2$ has no invariant subspaces other than $\{0\}$ and $\mathbf{R}^2$, therefore it is irreducible. However, its complexification, $\Pi^\dagger_{\text{nat}}$, has one-dimensional eigenspaces in $\mathbf{C}^2$, therefore it is reducible.

6 Schur's Lemma

Schur's Lemma [24,5,12] is a basic result in group representation theory. We supply a proof here to make the article self-contained and to motivate the variants that follow.

Lemma 1 (Schur's Lemma) *Suppose $\mathscr{X}$ is a finite-dimensional vector space over the complex field and Π is a representation on $\mathscr{X}$ of a group G. If Π is irreducible and if $A \in Lin(\mathscr{X})$ commutes with $\Pi(g)$ for all $g \in G$, then $A \in Sph(\mathscr{X})$, that is, A is a scalar multiple of the identity.*

[2] The group generated by a subset S of $Orth(\mathscr{X})$ is the smallest subgroup of $Orth(\mathscr{X})$ that contains S.

Proof Let α be an (possibly complex) eigenvalue of A. Define the linear operator $T = A - \alpha I$. Since A commutes with $\Pi(g)$ for every $g \in G$, it follows that,

$$T \circ \Pi(g) = \Pi(g) \circ T, \quad \forall g \in G. \tag{15}$$

Clearly the null space Null(T) of T is invariant under Π (this holds for null space of any linear operator on $\mathscr{X}$ that satisfies (15)). Since α is an eigenvalue of A, Null$(T) =$ Null$(A - \alpha I) \neq \{0\}$. Therefore, Null$(T) = \mathscr{X}$ since Π is an irreducible representation. Therefore $T = 0$, and thus $A = \alpha I$. □

The proof given above relies on the fact that any linear operator on a finite-dimensional complex vector space has an eigenvalue. Although this is not true in the case of *real* vector spaces, any supplementary condition that ensures the existence of a (real) eigenvalue will suffice for deducing the same conclusion. The following two variants of Schur's Lemma for *real* vector spaces provide the tools that we need. The first one is true because a symmetric linear transformation has real eigenvalues. The second one is true because linear transformations on odd-dimensional real vector spaces are guaranteed to have at least one real eigenvalue.

Lemma 2 *Suppose $\mathscr{X}$ is a finite-dimensional inner product space over the reals and Π is a representation on $\mathscr{X}$ of a group G. If Π is irreducible and if $A \in Sym(\mathscr{X})$ commutes with $\Pi(g)$ for all $g \in G$, then $A \in Sph(\mathscr{X})$, that is, A is a scalar multiple of the identity.*

Lemma 3 *Suppose $\mathscr{X}$ is a odd-dimensional vector space over the reals and Π is a representation on $\mathscr{X}$ of a group G. If Π is irreducible and if $A \in Lin(\mathscr{X})$ commutes with $\Pi(g)$ for all $g \in G$, then $A \in Sph(\mathscr{X})$, that is, A is a scalar multiple of the identity.*

The following lemma is in some sense a converse to Schur's Lemma.

Lemma 4 *Suppose G is a group with an orthogonal representation Π over a (real or complex) inner-product space $\mathscr{X}$. If for every $A \in Sym(\mathscr{X})$ we have:*

$$\Big\{A \circ \Pi(g) = \Pi(g) \circ A, \quad \forall g \in G\Big\} \Rightarrow A \in Sph(\mathscr{X}), \tag{16}$$

then Π is irreducible.

Proof Suppose to the contrary that Π is reducible. Then, there is a nontrivial subspace $U \subset \mathscr{X}$ which is invariant under Π. Since Π is an orthogonal representation, then $U^\perp$, the orthogonal complement of U, is invariant under Π as well. Let $P : \mathscr{X} \to \mathscr{X}$ be the orthogonal projection of $\mathscr{X}$ onto U. Pick any $\mathbf{v} \in \mathscr{X}$ and write $\mathbf{v} = \mathbf{u} + \mathbf{u}'$ where $\mathbf{u} \in U$ and $\mathbf{u}' \in U^\perp$. Then:

$$\begin{aligned} P \circ \Pi(g)(\mathbf{v}) = P \circ \Pi(g)(\mathbf{u} + \mathbf{u}') &= P \circ \Pi(g)(\mathbf{u}) + P \circ \Pi(g)(\mathbf{u}') \\ &= \Pi(g)(\mathbf{u}) = \Pi(g) \circ P(\mathbf{u} + \mathbf{u}') = \Pi(g) \circ P(\mathbf{v}). \end{aligned}$$

Thus $P \in Sym(\mathscr{X})$ and $P \circ \Pi(g) = \Pi(g) \circ P$ for every $g \in G$. However, P is not a multiple of identity, which contradicts (16). □

The following technical result is also closely related to Schur's Lemma.

Lemma 5 *Let G be a group with irreducible representations Π and Π' over finite-dimensional vector spaces $\mathscr{V}$ and $\mathscr{W}$, respectively. Let $A : \mathscr{V} \to \mathscr{W}$ be a linear map satisfying,*

$$A \circ \Pi(g) = \Pi'(g) \circ A, \quad \forall g \in G. \tag{17}$$

If $\dim \mathscr{V} \neq \dim \mathscr{W}$ *then* $A = 0$.

Proof If $\dim \mathscr{V} > \dim \mathscr{W}$, then $\mathrm{Null}(A) \neq \{0\}$. The identity (17) implies that $\mathrm{Null}(A)$ is invariant under Π. Therefore $\mathrm{Null}(A) = \mathscr{V}$ since Π is irreducible. If $\dim \mathscr{V} < \dim \mathscr{W}$, then $\mathrm{Range}(A)$ is a proper subspace of $\mathscr{W}$. Again, (17) implies that $\mathrm{Range}(A)$ is invariant under Π', and since Π' is irreducible, we have $\mathrm{Range}(A) = \{0\}$. □

7 The main theorems

Theorem 1 gives sufficient conditions on a symmetry group of an hyperelastic material that imply its isotropy. Theorem 2 states that if a group G fails to meets the conditions of Theorem 1, then there exists an anisotropic hyperelastic material with G as a symmetry group. Theorems 3 and 4 extend Theorem 1 to the non-hyperelastic and polar media, respectively, in three dimensions.

7.1 Hyperelasticity

Theorem 1 *Let $\mathscr{V}$ be the n-dimensional (real) Euclidean space and let* $\mathsf{C} : Sym \to Sym$ *be linear and self-adjoint. Let $G \subseteq Orth$ be such that the representations Π_{nat} and Π_{dev} defined in* (8) *and* (11) *are both irreducible. Then the following are equivalent:*

(a) *G is a symmetry group for* C*, that is,* (3) *holds.*
(b) C *has the representation in* (2)*.*
(c) C *is isotropic, that is* (1) *holds.*

Proof The implications (b) ⇒ (c) and (c) ⇒ (a) are obvious. It remains to show (a) ⇒ (b).

Letting $E = I$ in the identity (3) yields $C[I]Q = QC[I]$; that is, $C[I]$ commutes with every Q in G. Applying Lemma 2 with $A = \mathsf{C}[I]$ and $\Pi = \Pi_{\text{nat}}$, we conclude that $C[I]$ is a scalar multiple of identity. In particular, C leaves Sph invariant. Finally, since C is self-adjoint, it leaves $Sph^{\perp} = Dev$ invariant as well. Let $\mathsf{C}_{\text{sph}} : Sph \to Sph$ and $\mathsf{C}_{\text{dev}} : Dev \to Dev$ be the restrictions of C to Sph and Dev respectively.

The identity (3), restricted to $E \in Dev$ and expressed in terms of Π_{dev}, takes the form $\Pi_{\text{dev}}(Q)\big[\mathsf{C}[E]\big] = \mathsf{C}\big[\Pi_{\text{dev}}(Q)[E]\big]$. Therefore $\Pi_{\text{dev}}(Q) \circ \mathsf{C}_{\text{dev}} = \mathsf{C}_{\text{dev}} \circ \Pi_{\text{dev}}(Q)$ for all $Q \in G$. Since C_{dev} is self-adjoint, the irreducibility of Π_{dev} along with Lemma 2 implies that $\mathsf{C}_{\text{dev}} \in Sph(Dev)$; that is, C_{dev} is a scalar multiple of the identity map on Dev.

According to the decomposition in (5), any $E \in Sym$ is of the form $E = E_{\text{sph}} + E_{\text{dev}}$ where $E_{\text{sph}} \in Sph$ and $E_{\text{dev}} \in Dev$. Therefore

$$\mathsf{C}[E] = \mathsf{C}[E_{\text{sph}} + E_{\text{dev}}] = \mathsf{C}_{\text{sph}}[E_{\text{sph}}] + \mathsf{C}_{\text{dev}}[E_{\text{dev}}] = \alpha E_{\text{sph}} + \beta E_{\text{dev}}$$

for some real constants α and β. But $E_{\text{sph}} = \frac{1}{n}(\operatorname{tr} E)I$ and $E_{\text{dev}} = E - E_{\text{sph}}$. Therefore

$$\mathsf{C}[E] = \frac{\alpha}{n}(\operatorname{tr} E)I + \beta\Big(E - \frac{1}{n}(\operatorname{tr} E)I\Big) = \beta E + \frac{\alpha - \beta}{n}(\operatorname{tr} E)I = 2\mu E + \lambda(\operatorname{tr} E)I,$$

with $2\mu = \alpha/n$ and $\lambda = (\alpha - \beta)/n$. □

Consider an elasticity tensor C, and suppose G is a symmetry group of C. Theorem 1 says that if the natural and deviatoric representations of G are irreducible then C is isotropic. The following provides a converse.

Theorem 2 *Let G be a subgroup of Orth. If for all linear and self-adjoint mappings* $\mathsf{C} : Sym \to Sym$ *we have*

$$\left\{ \mathsf{C}[QEQ^T] = Q\mathsf{C}[E]Q^T, \quad \forall Q \in G,\ E \in Sym \right\} \Rightarrow \mathsf{C} \text{ has the representation in } (2)\ , \quad (18)$$

then the natural and deviatoric representations of G are irreducible.

Proof We prove the statement for Π_{dev}, the same argument applies to Π_{nat}. Let $\mathsf{D} : Dev \to Dev$, be linear and self-adjoint, and suppose that $\mathsf{D} \circ \Pi_{\text{dev}}(Q) = \Pi_{\text{dev}}(Q) \circ \mathsf{D}$ for every $Q \in G$. It follows from (18) that $\mathsf{D} \in Sph(Dev)$. This shows that

$$\left\{ \mathsf{D} \circ \Pi_{\text{dev}}(Q) = \Pi_{\text{dev}}(Q) \circ \mathsf{D}, \quad \forall Q \in G \right\} \Rightarrow \mathsf{D} \in Sph(Dev).$$

Hence, it follows from Lemma 4 that Π_{dev} is irreducible. □

Remark 4 The assumption of self-adjointness in Theorem 1 may be dropped provided that we replace the assumption of irreducibility of Π_{dev} by the irreducibility of its *complexification*, $\Pi^{\dagger}_{\text{dev}}$ on $Dev^{\dagger}$. See section 9.3 for an application of this idea.

7.2 Three-dimensional elasticity in the non-hyperelastic case

In three-dimensional elasticity, we have $\dim \mathscr{V} = 3$ and $\dim Dev = 5$. Therefore, in view of Lemma 3, the assumption of C being self-adjoint in the statement of Theorem 1 may be dropped, thus extending it to non-hyperelastic materials. We record this interesting result in the following theorem:

Theorem 3 *Let* $\mathscr{V}$ *be the three-dimensional Euclidean vector space and let* $\mathsf{C} : Sym \to Sym$ *be linear and not necessarily self-adjoint. Let* $G \subseteq Orth$ *be such that the representations* Π_{nat} *and* Π_{dev} *defined in* (8) *and* (11) *are both irreducible. Then the following are equivalent:*

(a) *G is a symmetry group for* C*, that is,* (3) *holds.*
(b) C *has the representation in* (2)*.*
(c) C *is isotropic, that is* (1) *holds.*

Proof As before, it suffices to prove that (a) ⇒ (b). Letting $E = I$ in the identity (3) yields $C[I]Q = QC[I]$; that is, $C[I]$ commutes with every Q in G. Applying Lemma 2 with $A = \mathsf{C}[I]$ and $\Pi = \Pi_{\text{nat}}$, we conclude that $C[I]$ is a scalar multiple of identity, and thus C leaves Sph invariant. In view of (14), the identity (3) also holds for the adjoint C^T of C. Hence, applying the same argument to C^T, we conclude that C^T leaves Sph invariant as well. Moreover, since for every $E \in Dev$ we have $\langle\langle \mathsf{C}[E], I \rangle\rangle = \langle\langle E, \mathsf{C}^T[I] \rangle\rangle = 0$, then C leaves Dev invariant.

As in the proof of Theorem 1, the identity (3), restricted to $E \in Dev$ and expressed in terms of Π_{dev}, implies $\Pi_{\text{dev}}(Q) \circ \mathsf{C}_{\text{dev}} = \mathsf{C}_{\text{dev}} \circ \Pi_{\text{dev}}(Q)$ for all $Q \in G$. Then, irreducibility of Π_{dev} along with Lemma 3 imply that $\mathsf{C}_{\text{dev}} \in Sph(Dev)$. The rest of the proof is the same as that of Theorem 1. □

7.3 Polar elasticity in three dimensions

In this section we extend the previous theory to polar media where the strain and stress are no longer symmetric, thus the elasticity tensor is replaced with a linear mapping $\mathsf{C} : \mathit{Lin} \to \mathit{Lin}$. The foundations of polar media are described in [29, section 98]. Also see [20] for a survey and review, and [21, chapter 6] for a very accessible elementary account. The main result of this section is:

Theorem 4 *Let $\mathscr{V}$ be the three-dimensional Euclidean vector space and let $\mathsf{C} : \mathit{Lin} \to \mathit{Lin}$ be linear and not necessarily self-adjoint. Let $G \subseteq \mathit{Orth}$ be a group and suppose*

$$Q\mathsf{C}[A]Q^T = \mathsf{C}[QAQ^T] \quad \text{for all } A \in \mathit{Lin},\ Q \in G. \tag{19}$$

If the representations Π_{nat}, Π_{dev} and Π_{skw} of G are irreducible, then

$$\mathsf{C}[A] = a(\text{tr}A)I + bA + cA^T, \quad \text{for all } A \in \mathit{Lin}$$

where a, b, and c are real constants.

We will provide a proof of Theorem 4 in Section 7.3.3, after establishing the prerequisite framework.

7.3.1 Linear transformations on Skew

Here we focus on linear mappings $\mathsf{W} : \mathit{Skew} \to \mathit{Skew}$. The following result shows the condition under which such mappings are a multiple of identity.

Theorem 5 *Let $\mathscr{V}$ be the three-dimensional Euclidean vector space and let $\mathsf{W} : \mathit{Skew} \to \mathit{Skew}$ be linear and not necessarily self-adjoint. Let $G \subseteq \mathit{Orth}$ be a group and suppose*

$$Q\mathsf{W}[A]Q^T = \mathsf{W}[QAQ^T] \quad \text{for all } A \in \mathit{Skew},\ Q \in G. \tag{20}$$

If the representation Π_{skw} of G is irreducible, then $\mathsf{W} \in \mathit{Sph}(\mathit{Skew})$, that is, there exists a real constant ξ such that $\mathsf{W}[A] = \xi A$ for all $A \in \mathit{Skew}$.

Proof The identity (20) is equivalent to

$$\Pi_{\text{skw}}(Q) \circ \mathsf{W} = \mathsf{W} \circ \Pi_{\text{skw}}(Q) \quad \text{for all } Q \in G,$$

Since *Skew* is odd-dimensional ($\dim \mathit{Skew} = 3$), Lemma 3 implies that W is a scalar multiple of identity. ☐

7.3.2 Linear transformations on Lin

Our next goal is to split a linear map $\mathsf{C} : \mathit{Lin} \to \mathit{Lin}$ into the sum of three operators conforming to the decomposition

$$\mathit{Lin} = \mathit{Sph} \oplus \mathit{Dev} \oplus \mathit{Skew}.$$

See Proposition 1 for the precise statement.

Lemma 6 *Let $G \subseteq \mathit{Orth}$ be a group. Let $\mathscr{V}$ be the three-dimensional Euclidean vector space and $\mathsf{C} : \mathit{Lin} \to \mathit{Lin}$ be a linear map for which the identity (19) holds. If the natural representation Π_{nat} of G is irreducible, then C leaves Sph and $\mathit{Dev} \oplus \mathit{Skew}$ invariant.*

Proof Letting $A = I$ in (19) gives,

$$\mathsf{C}[I]Q = Q\mathsf{C}[I], \quad \text{for all } Q \in G, \tag{21}$$

Therefore, Lemma 3 implies that $\mathsf{C}[I] = \alpha I$ for some $\alpha \in \mathbf{R}$; thus, C leaves *Sph* invariant. An analogous argument applied to the adjoint C^T of C shows that C^T leaves *Sph* invariant. Moreover, for any $A \in Dev \oplus Skew$ we have $\langle\langle \mathsf{C}[A], I\rangle\rangle = \langle\langle A, C^T[I]\rangle\rangle = 0$; thus, C leaves $Dev \oplus Skew$ invariant. □

In view of the above lemma, we decompose the action of C as $\mathsf{C}[A] = \mathsf{C}_0[A] + \mathsf{C}_1[A]$ for all $A \in Lin$, where C_0 and C_1 are the restrictions of C to *Sph* and $Dev \oplus Skew$, respectively. We further decompose C_1 by defining $\mathsf{C}^s : Dev \oplus Skew \to Dev$ and $\mathsf{C}^w : Dev \oplus Skew \to Skew$ via

$$\mathsf{C}^s[A] = \frac{1}{2}\big(\mathsf{C}_1[A] + \mathsf{C}_1[A]^T\big), \quad \mathsf{C}^w[A] = \frac{1}{2}\big(\mathsf{C}_1[A] - \mathsf{C}_1[A]^T\big), \quad A \in Dev \oplus Skew. \tag{22}$$

To fully decouple the action of C, we need to show that $\mathsf{C}^s[A] = 0$ for every $A \in Skew$ and $\mathsf{C}^w[A] = 0$ for every $A \in Dev$.

Lemma 7 *Let $G \subseteq Orth$ be a group. Let $\mathscr{V}$ be the three-dimensional Euclidean vector space and $\mathsf{C} : Lin \to Lin$ be a linear map for which the identity* (19) *holds. If the deviatoric and skew representations, Π_{dev} and Π_{skw}, of G are irreducible, then*

1. $\mathsf{C}^s[A] = 0$ *for all $A \in Skew$,*
2. $\mathsf{C}^w[A] = 0$ *for all $A \in Dev$.*

Proof We demonstrate the prove of the first statement since that of the second statement is similar.

We have for all A in $Dev \oplus Skew$ and $Q \in G$

$$Q\mathsf{C}^s[A]Q^T = \frac{1}{2}\big(Q\mathsf{C}_1[A]Q^T + Q\mathsf{C}_1[A]^T Q^T\big) = \frac{1}{2}\big(\mathsf{C}_1[QAQ^T] + \mathsf{C}_1[QAQ^T]^T\big) = \mathsf{C}^s[QAQ^T].$$

Letting $\mathsf{C}^s_w : Skew \to Dev$ be the restriction of C^s to *Skew*, this implies that $Q\mathsf{C}^s_w[A]Q^T = \mathsf{C}^s_w[QAQ^T]$ for all $A \in Skew$ and $Q \in G$; or equivalently,

$$\Pi_{\text{dev}}(Q)\big[\mathsf{C}^s_w[A]\big] = \mathsf{C}^s_w\big[\Pi_{\text{skw}}(Q)[A]\big], \quad \forall A \in Skew, Q \in G,$$

that is,

$$\Pi_{\text{skw}}(Q) \circ \mathsf{C}^s_w = \mathsf{C}^s_w \circ \Pi_{\text{dev}}(Q) \quad \text{for all } Q \in G.$$

Since the dimensions of *Skew* and *Dev* are different ($\dim Skew = 3$ and $\dim Dev = 5$), and the representations Π_{dev} and Π_{skw} of G are irreducible, it follows from Lemma 5 that $\mathsf{C}^s_w = 0$. □

Let π_{Sph}, π_{Dev}, and π_{Skew} be orthogonal projections of *Lin* onto *Sph*, *Dev*, and *Skew*, respectively; that is, for any $A \in Lin$,

$$\pi_{Sph}(A) = \frac{1}{3}(\text{tr}A)I, \quad \pi_{Dev}(A) = \frac{1}{2}(A + A^T) - \frac{1}{3}(\text{tr}A)I, \quad \pi_{Skew}(A) = \frac{1}{2}(A - A^T).$$

Let $\mathsf{D} : Dev \to Dev$ be the restriction of C^s to *Dev* and $\mathsf{W} : Skew \to Skew$ be the restriction of C^w to *Skew*. By combining Lemmas 6 and 7, we arrive at the following decomposition result:

Proposition 1 *Let $G \subseteq Orth$ be a group. Let $\mathscr{V}$ be the three-dimensional Euclidean vector space and $\mathsf{C} : Lin \to Lin$ be a linear map for which the identity* (19) *holds. If the representations Π_{nat}, Π_{dev}, and Π_{skw} of G are irreducible, then*

$$\mathsf{C} = \mathsf{C}_0 \circ \pi_{Sph} + \mathsf{D} \circ \pi_{Dev} + \mathsf{W} \circ \pi_{Skew}. \tag{23}$$

Remark 5 The significance of the Proposition 1 is that a linear mapping $\mathsf{C} : Lin \to Lin$ that satisfies its hypotheses, leaves the subspaces *Sph*, *Dev*, and *Skew* of *Lin* invariant.

7.3.3 Proof of Theorem 4

Let us note that under the hypothesis of the Theorem, Proposition 1 applies. By Lemma 6, $\mathsf{C}_0[A] = \xi_0 A$ for all $A \in Sph$, where ξ_0 is a real constant. From (19) we have that $Q\mathsf{D}[A]Q^T = \mathsf{D}[QAQ^T]$ for all $A \in Dev$ and $Q \in G$; or equivalently:

$$\mathsf{D} \circ \Pi_{\text{dev}}(Q) = \Pi_{\text{dev}}(Q) \circ \mathsf{D}, \quad \text{for all } Q \in G.$$

Thus, using irreducibility of Π_{dev} and the fact that *Dev* is odd-dimensional ($\dim Dev = 5$) we apply Lemma 3 to get that $\mathsf{D}[A] = \xi_1 A$ for all $A \in Dev$ where ξ_1 is a real constant.

Next, we note that (19) implies that $Q\mathsf{W}[A]Q^T = \mathsf{W}[QAQ^T]$ for all $A \in Skew$ and $Q \in G$. Thus, Theorem 5 gives $\mathsf{W}[A] = \xi_2 A$ for all $A \in Skew$ where ξ_2 is a real constant. Applying the decomposition (23) we have for an arbitrary $A \in Lin$,

$$\begin{aligned}
\mathsf{C}[A] &= \mathsf{C}_0[\pi_{Sph}(A)] + \mathsf{D}[\pi_{Dev}(A)] + \mathsf{W}[\pi_{Skew}(A)] \\
&= \xi_0 \pi_{Sph}(A) + \xi_1 \pi_{Dev}(A) + \xi_2 \pi_{Skew}(A) \\
&= \xi_0 \Big(\frac{1}{3}(\text{tr}A)I\Big) + \xi_1 \Big(\frac{A + A^T}{2} - \frac{1}{3}(\text{tr}A)I\Big) + \xi_2 \Big(\frac{A - A^T}{2}\Big) \\
&= a(\text{tr}A)I + bA + cA^T,
\end{aligned}$$

with $a = (\xi_0 - \xi_1)/3$, $b = (\xi_1 + \xi_2)/2$, and $c = (\xi_1 - \xi_2)/2$. □

8 The classical representation theorem for isotropic elasticity tensors

Throughout this section, $\mathscr{V}$ is the n-dimensional Euclidean vector space. We derive the classical representation (2) for hyperelastic materials as an application of Theorem 1 with the symmetry group $G = Orth$. For this, we need to verify that when $G = Orth$, the representations Π_{nat} and Π_{dev} are irreducible. The irreducibility of Π_{dev} is a consequence of the following lemma which is implied by a lemma in [19]:

Lemma 8 *Let $E \in Dev$, $E \neq 0$ and $F \in Dev$. If $\langle\langle \Pi_{\text{dev}}(Q)[E], F \rangle\rangle = 0$ for all $Q \in Orth$, then $F = 0$.*

With the aid of this, we have:

Theorem 6 *If the symmetry group G in* (3) *is the orthogonal group Orth, then* (2) *holds.*

Proof When $G = Orth$, the representations Π_{nat} is clearly irreducible because the only subspaces of $\mathscr{V}$ that are invariant under all rotations are $\{0\}$ and $\mathscr{V}$. To show the irreducibility of Π_{dev}, suppose U is a nontrivial subspace of *Dev* which is invariant under Π_{dev}. Then $\Pi_{\text{dev}}(Q)U = U$ for all $Q \in Orth$. Choose a nonzero E in U and let $F \in U^\perp \subset Dev$. Then $\langle\langle \Pi_{\text{dev}}(Q)[E], F \rangle\rangle = 0$ for all $Q \in Orth$. From Lemma 8 it follows that $F = 0$ therefore $U = Dev$ and thus the representation Π_{dev} of *Orth* is irreducible. Then the assertion of the lemma follows from Theorem 1. □

9 Isotropy in two-dimensional elasticity

In two-dimensional hyperelasticity, if the identity (3) holds for a rotation Q, then the material is isotropic provided that Q's rotation angle is other than an integer multiple of $\pi/2$. This assertion, stated formally in Theorem 7 below, follows from Theorem 1 with the aid of two elementary lemmas.

In the non-hyperelastic case, addressed in Theorem 8, if the identity (3) holds for a rotation by an angle other than an integer multiple of $\pi/2$ and also a reflection, then the elasticity tensor is isotropic. Throughout this section, $\mathscr{V}$ is the two-dimensional Euclidean vector space.

Remark 6 When we speak of a symmetry group in the context of two-dimensional elasticity, we consider the material as two-dimensional mathematical object, and not the two-dimensional cross-section of a three-dimensional material. The isotropy of a two-dimensional elastic material is a statement about the independence of orientation within its plane; the third dimension does not enter the picture. Two-dimensional cases arise naturally in plane strain or plane stress problems.

9.1 Reflections and rotations in two-dimensions

Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be an orthonormal basis of $\mathscr{V}$, and let R be a reflection about $\mathbf{e}_1$; that is, $R\mathbf{e}_1 = \mathbf{e}_1$ and $R\mathbf{e}_2 = -\mathbf{e}_2$. Then, R has eigenvalues $+1$ and -1 corresponding to eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$. Let Q be a rotation of $\mathscr{V}$ by an angle θ. Then we have,

$$Q\mathbf{e}_1 = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2, \quad Q\mathbf{e}_2 = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2.$$

On the complexification $\mathscr{V}^\dagger$ of $\mathscr{V}$, Q has eigenvalues $e^{\pm i\theta}$ corresponding to eigenvectors $\mathbf{e}_1 \mp i\mathbf{e}_2$.

9.2 The hyperelastic case

Our goal here is to apply Theorem 1 to a group generated by a single rotation on $\mathscr{V}$. The condition on irreducibility of the natural representation is addressed by the following lemma.

Lemma 9 *Let Q be a rotation of $\mathscr{V}$ by an angle θ which is other than an integer multiple of π, and let G be the subgroup of Orth generated by Q. Then the representation Π_{nat} of G is irreducible.*

Proof Let U be a nontrivial subspace of $\mathscr{V}$ which is invariant under the representation Π_{nat}. Then $\Pi_{\text{nat}}(Q)U = QU \subset U$. Now, for any nonzero vector $\mathbf{v} \in U$, the vectors $\mathbf{v}$ and $Q\mathbf{v}$ are linearly independent because θ is not an integer multiple of π. Hence U is two-dimensional, therefore $U = \mathscr{V}$. Consequently, Π_{nat} has no invariant subspaces other than $\{0\}$ and $\mathscr{V}$ and thus it is irreducible. □

Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be the orthonormal basis of $\mathscr{V}$ as above. It is straightforward to verify that the complexified operator $\Pi^\dagger_{\text{dev}}(Q) : Dev^\dagger \to Dev^\dagger$ has eigenvalues $e^{\mp 2i\theta}$ corresponding to the eigenvectors $E_1 \pm iE_2$, where:

$$E_1 = \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2, \quad E_2 = \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1, \tag{24}$$

and $\{E_1, E_2\}$ is an orthogonal basis of $Dev^{\dagger}$. When θ is not an integer multiple of $\pi/2$, the eigenvalues are complex, therefore the mapping $\Pi_{\mathrm{dev}}(Q)$ has no one-dimensional invariant subspaces. This leads to the following lemma.

Lemma 10 *Let Q be a rotation of $\mathscr{V}$ by an angle θ which is other than an integer multiple of $\pi/2$ and let G be the subgroup of Orth generated by Q. Then the representation Π_{dev} of G is irreducible.*

Proof Under the assumptions, $\Pi_{\mathrm{dev}}(Q)$ has no real eigenvalues, therefore it cannot have a one-dimensional invariant subspace. Hence, if $U \subseteq Dev$ is a subspace invariant under Π_{dev}, then either $U = \{0\}$ or $U = Dev$. $\square$

Combining the assertions of the preceding two lemmas with Theorem 1 we arrive at:

Theorem 7 *Let $\mathscr{V}$ be the two-dimensional Euclidean vector space and let $\mathsf{C} : Sym \to Sym$ be linear and self-adjoint. Suppose that the identity*

$$Q\mathsf{C}[E]Q^T = \mathsf{C}[QEQ^T] \quad \textit{for all } E \in Sym,$$

holds for a particular $Q \in Orth$ that represents a rotation by a certain angle θ. If θ is not an integer multiple of $\pi/2$, then C is isotropic.

9.3 The non-hyperelastic case

If we lift the assumption of C being self-adjoint, Theorem 7 no longer holds. In this case we recall Remark 4 which allows for lifting the assumption of self-adjointness of C in Theorem 1 at the expense of checking irreducibility of $\Pi^{\dagger}_{\mathrm{dev}}$.

The representation $\Pi^{\dagger}_{\mathrm{dev}}$ of a group generated by a rotation Q is not irreducible on $Dev^{\dagger}$ as $\Pi^{\dagger}_{\mathrm{dev}}(Q)$ has one-dimensional eigenspaces:

$$\mathrm{span}\{E_1 + iE_2\}, \quad \mathrm{span}\{E_1 - iE_2\},$$

where E_1 and E_2 are as in (24). Let R be a reflection of $\mathscr{V}$ about axis $\mathbf{e}_1$ as in Section 9.1. It is evident that $\Pi_{\mathrm{dev}}(R)[E_1] = E_1$ and $\Pi_{\mathrm{dev}}(R)[E_2] = -E_2$; that is, $\Pi_{\mathrm{dev}}(R)$ has eigenvalues ± 1 corresponding to eigenvectors E_1 and E_2.

Let Q be a rotation with angle θ which is not an integer multiple of $\pi/2$ and R a reflection. By the above discussion, the invariant subspaces of $\Pi^{\dagger}_{\mathrm{dev}}(R)$ in $Dev^{\dagger}$ are:

$$\{0\}, \mathrm{span}\{E_1\}, \mathrm{span}\{E_2\}, Dev^{\dagger},$$

and the invariant subspaces of $\Pi^{\dagger}_{\mathrm{dev}}(Q)$ in $Dev^{\dagger}$ are:

$$\{0\}, \mathrm{span}\{E_1 + iE_2\}, \mathrm{span}\{E_1 - iE_2\}, Dev^{\dagger}.$$

The only subspaces of $Dev^{\dagger}$ which are invariant under both $\Pi^{\dagger}_{\mathrm{dev}}(R)$ and $\Pi^{\dagger}_{\mathrm{dev}}(Q)$ are $\{0\}$ and $Dev^{\dagger}$. We have just proved the following:

Lemma 11 *Let R be a reflection in $\mathscr{V}$ and let Q be a rotation of $\mathscr{V}$ by an angle θ which is other than an integer multiple of $\pi/2$. Let G be the subgroup of Orth generated by R and Q. Then the representation $\Pi^{\dagger}_{\mathrm{dev}}$ of G over $Dev^{\dagger}$ is irreducible.*

Clearly the natural representation of the group G described in Lemma 11 is irreducible also. Therefore, Lemma 11 along with Remark 4 gives the following result:

Theorem 8 *Let $\mathscr{V}$ be the two-dimensional Euclidean vector space and let $\mathsf{C} : Sym \to Sym$ be linear and not necessarily self-adjoint. Suppose that the identities*

$$R\mathsf{C}[E]R^T = \mathsf{C}[RER^T] \quad \textit{and} \quad Q\mathsf{C}[E]Q^T = \mathsf{C}[QEQ^T] \quad \textit{for all } E \in Sym,$$

hold, where R is a reflection and Q is a rotation of $\mathscr{V}$ by an angle θ other than an integer multiple of $\pi/2$. Then C is isotropic.

9.4 Polar elasticity

Theorem 8 generalizes to polar media as follows:

Theorem 9 *Let $\mathscr{V}$ be the two-dimensional Euclidean vector space and let $\mathsf{C} : Lin \to Lin$ be linear and not necessarily self-adjoint. Suppose that the identities*

$$R\mathsf{C}[A]R^T = \mathsf{C}[RAR^T] \quad \textit{and} \quad Q\mathsf{C}[A]Q^T = \mathsf{C}[QAQ^T] \quad \textit{for all } A \in Lin,$$

hold, where R is a reflection and Q is a rotation of $\mathscr{V}$ by an angle θ other than an integer multiple of $\pi/2$. Then, there exist real constants a, b, c such that

$$\mathsf{C}[A] = a(\mathrm{tr}A)I + bA + cA^T, \quad \textit{for all } A \in Lin.$$

The proof of this, which we omit to avoid repetition, is along the lines of the arguments presented in Theorem 4. An observation that enters the proof is that the complexified representation $\Pi^{\dagger}_{\mathrm{nat}}$ is irreducible on $\mathscr{V}^{\dagger}$.

10 Isotropy in three-dimensional elasticity

One of the interesting results in [11] is that elastic symmetry under two generic rotations in three dimensions implies isotropy. (The term "generic" is made precise in Definition 1 below.) We rederive that result here as an application of our Theorem 3. This requires verifying the theorem's irreducibility hypotheses which involves lengthy and detailed calculations. We rely on the analysis in [11] for some of the technical details.

10.1 Rotations in three dimension

Throughout this section, $\mathscr{V}$ is the 3-dimensional Euclidean vector space. Let us denote by $R^{\alpha}_{\mathbf{u}}$ a rotation by angle α about a unit vector $\mathbf{u} \in \mathscr{V}$. The composition of two rotations $R^{\alpha}_{\mathbf{u}}$ and $R^{\beta}_{\mathbf{v}}$ is rotation $R^{\gamma}_{\mathbf{w}}$, where $\mathbf{w}$ and γ can be computed as follows (see the Appendix in [11] for an algebraic proof). Let

$$Y = \cos\frac{\alpha}{2}\cos\frac{\beta}{2} - (\mathbf{u}\cdot\mathbf{v})\sin\frac{\alpha}{2}\sin\frac{\beta}{2}. \tag{25}$$

Then,

$$\mathbf{w} = \frac{\operatorname{sign}(Y)}{\sqrt{1-Y^2}}\Big[\big(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\big)\mathbf{u} + \big(\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\big)\mathbf{v} + \big(\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\big)\mathbf{u}\times\mathbf{v}\Big], \tag{26a}$$

$$\cos\gamma = -1 + Y^2, \tag{26b}$$

$$\sin\gamma = 2|Y|\sqrt{1-Y^2}. \tag{26c}$$

In (26a), $\mathbf{u}\times\mathbf{v}$ denotes the familiar cross product of $\mathbf{u}$ and $\mathbf{v}$.

10.2 Symmetry groups generated by two rotations

Following [11], we employ the notation

$$[\alpha,\beta,d] = \{G \subseteq Orth:\ \text{G is the group generated by } R_{\mathbf{u}}^{\alpha}, R_{\mathbf{v}}^{\beta} \text{ with } \mathbf{u}\cdot\mathbf{v} = d\}.$$

Thus $[\alpha,\beta,d]$ is a collection of subgroups of *Orth*, and if G_1 and G_2 are in $[\alpha,\beta,d]$, then there exists $Q \in Orth$ such that $G_1 = QG_2Q^T$.

Let $R_{\mathbf{u}}^{\alpha}$, $R_{\mathbf{v}}^{\beta}$, and $R_{\mathbf{w}}^{\gamma}$ be three rotations such that $R_{\mathbf{u}}^{\alpha}R_{\mathbf{v}}^{\beta} = R_{\mathbf{w}}^{\gamma}$. Then any group containing two of the rotations will also include the third; moreover, any two of $R_{\mathbf{u}}^{\alpha}$, $R_{\mathbf{v}}^{\beta}$ and $R_{\mathbf{w}}^{\gamma}$ generate the same group and $[\alpha,\beta,\mathbf{u}\cdot\mathbf{v}] = [\alpha,\gamma,\mathbf{u}\cdot\mathbf{w}] = [\beta,\gamma,\mathbf{v}\cdot\mathbf{w}]$.

10.3 Representations of symmetry groups generated by two rotations

Let $\mathbf{u}$ and $\mathbf{v}$ be a pair of non-collinear vectors in $\mathscr{V}$ and let us consider the group G generated by the rotations $R_{\mathbf{u}}^{\alpha}$ and $R_{\mathbf{v}}^{\beta}$. The goal of this section is to investigate the irreducibility of the representations Π_{nat} and Π_{dev} of G. Noting that $R_{\mathbf{u}}^{\alpha} = R_{\mathbf{u}}^{\alpha+2\pi} = R_{-\mathbf{u}}^{-\alpha}$ and $R_{\mathbf{u}}^{0} = I$, we may confine the angles α and β to the interval $(0,\pi]$. We divide the discussion into the following mutually disjoint cases that cover all possibilities:

$$\left.\begin{array}{l} C_1 : \alpha \neq \pi \text{ and } \beta \neq \pi \\ C_2 : \alpha \neq \pi, \beta = \pi \text{ (or vice versa), and } \mathbf{u}\cdot\mathbf{v} \neq 0 \\ C_3 : \alpha \neq \pi, \beta = \pi \text{ (or vice versa), and } \mathbf{u}\cdot\mathbf{v} = 0 \\ C_4 : \alpha = \beta = \pi \end{array}\right\} \tag{27}$$

10.3.1 Case C_1

The groups corresponding to the case C_1 have irreducible natural representations, as proven in the following proposition.

Proposition 2 *Assume case C_1 of* (27) *holds. Then the natural representation Π_{nat} of the group G generated by $R_{\mathbf{u}}^{\alpha}$ and $R_{\mathbf{v}}^{\beta}$ is irreducible.*

Proof Let $U = \operatorname{span}\{\mathbf{u}\}$ and $V = \operatorname{span}\{\mathbf{v}\}$. Since $\alpha \neq \pi$ and $\beta \neq \pi$, then the only invariant subspaces of $R_{\mathbf{u}}^{\alpha}$ are $\big\{\{0\}, U, U^{\perp}, \mathscr{V}\big\}$ and the only invariant subspaces of $R_{\mathbf{v}}^{\beta}$ are $\big\{\{0\}, V, V^{\perp}, \mathscr{V}\big\}$. Thus if $R_{\mathbf{u}}^{\alpha}$ and $R_{\mathbf{v}}^{\beta}$ have a common nontrivial invariant proper subspace, then dimensional considerations imply that $U = V$ which contradicts the standing assumption that $\mathbf{u}$ and $\mathbf{v}$ are non-collinear. $\square$

The verification of irreducibility of the deviatoric representation is quite nontrivial. In the proof of the following result, we appeal to related analysis in [11].

Proposition 3 *Assume case C_1 of (27) holds. The deviatoric representation Π_{dev} of the group G generated by $R_{\mathbf{u}}^{\alpha}$ and $R_{\mathbf{v}}^{\beta}$ is irreducible if none of the following holds:*

1. $\alpha = \beta = \pi/2$ *and* $\mathbf{u}\cdot\mathbf{v} = 0$
2. $\alpha = \pi/2, \beta = 2\pi/3$ *(or vice versa) and* $|\mathbf{u}\cdot\mathbf{v}| = 1/\sqrt{3}$
3. $\alpha = \beta = 2\pi/3$ *and* $|\mathbf{u}\cdot\mathbf{v}| = 1/3$.

Proof According to Theorem 8.5 in [11], the complexifications $\Pi_{\text{dev}}^{\dagger}(R_{\mathbf{u}}^{\alpha})$ and $\Pi_{\text{dev}}^{\dagger}(R_{\mathbf{v}}^{\beta})$ share a common nontrivial invariant proper subspace of $Dev^{\dagger}$ only if at least one of the conditions 1–3 hold. Therefore, if none of the conditions hold, then $\Pi_{\text{dev}}^{\dagger}(R_{\mathbf{u}}^{\alpha})$ and $\Pi_{\text{dev}}^{\dagger}(R_{\mathbf{v}}^{\beta})$ share no common nontrivial invariant proper subspace, from which it follows that their real counterparts, $\Pi_{\text{dev}}(R_{\mathbf{u}}^{\alpha})$ and $\Pi_{\text{dev}}(R_{\mathbf{v}}^{\beta})$, share no common nontrivial invariant proper subspace, hence Π_{dev} is irreducible. □

10.3.2 Case C_2

As noted in [11], case C_2 provides no symmetry groups other than those in case C_1. Thus one obtains irreducibility of natural and deviatoric representations corresponding to case C_2 as a consequence of the corresponding results in case C_1. Let us show that any group in case C_2 corresponds to a group in C_1.

Lemma 12 *Let $R_{\mathbf{u}}^{\alpha}$ and $R_{\mathbf{v}}^{\pi}$ be rotations with $\alpha \in (0,\pi)$ and $\mathbf{u}\cdot\mathbf{v} \neq 0$, and let G be the group generated by $R_{\mathbf{u}}^{\alpha}$ and $R_{\mathbf{v}}^{\pi}$. Then, there exists a rotation $R_{\mathbf{w}}^{\gamma}$ with $\gamma \neq \pi$ such that G is generated by $R_{\mathbf{u}}^{\alpha}$ and $R_{\mathbf{w}}^{\gamma}$ also.*

Proof Let $R_{\mathbf{w}}^{\gamma} = R_{\mathbf{u}}^{\alpha} R_{\mathbf{v}}^{\pi}$. The vector $\mathbf{w}$ can be computed using (26a), and is clearly linearly independent from $\mathbf{u}$. It is now sufficient to show that $\gamma \neq \pi$. Note that by (25) we have $Y = -(\mathbf{u}\cdot\mathbf{v})\sin\frac{\alpha}{2} \neq 0$, and thus from (26b) we obtain $\cos\gamma = -1 + Y^2 > -1$. Therefore, $\gamma \neq \pi$. □

Using the composition formula in (26a), (26b),(26c) it is also possible to compute which groups of case C_2 correspond to the special groups in C_1 listed in Proposition 3. This correspondence was derived in [11] and is restated in the following:

Proposition 4 *The following identities hold:*

$$\left[\frac{2\pi}{3},\pi,\frac{1}{\sqrt{3}}\right] = \left[\frac{2\pi}{3},\pi,-\frac{1}{\sqrt{3}}\right] = \left[\frac{2\pi}{3},\frac{2\pi}{3},\frac{1}{3}\right] = \left[\frac{2\pi}{3},\frac{2\pi}{3},-\frac{1}{3}\right],$$

$$\left[\frac{\pi}{2},\pi,\frac{1}{\sqrt{2}}\right] = \left[\frac{\pi}{2},\pi,-\frac{1}{\sqrt{2}}\right] = \left[\frac{2\pi}{3},\pi,\sqrt{\frac{2}{3}}\right] = \left[\frac{2\pi}{3},\pi,-\sqrt{\frac{2}{3}}\right]$$

$$= \left[\frac{\pi}{2},\frac{\pi}{2},0\right] = \left[\frac{\pi}{2},\frac{2\pi}{3},\frac{1}{\sqrt{3}}\right] = \left[\frac{\pi}{2},\frac{2\pi}{3},-\frac{1}{\sqrt{3}}\right].$$

We return to our discussion of irreducibility of natural and deviatoric representations. Since Π_{nat} is irreducible for groups corresponding to case C_1, we know that the same holds for the case of C_2. The following proposition gives an alternative proof for case C_2 which is direct and self-contained.

Proposition 5 *Assume case C_2 of* (27) *holds. Then the natural representation Π_{nat} of the group G generated by $R^{\alpha}_{\mathbf{u}}$ and $R^{\beta}_{\mathbf{v}}$ is irreducible.*

Proof Let $U = \text{span}\{\mathbf{u}\}$ and $V = \text{span}\{\mathbf{v}\}$. Suppose $\alpha \neq \pi$ and $\beta = \pi$. The only invariant subspaces of $R^{\alpha}_{\mathbf{u}}$ are $\{\{0\}, U, U^{\perp}, \mathscr{V}\}$. The invariant subspaces of $R^{\beta}_{\mathbf{v}}$ consist of

$$\{0\}, \quad V, \quad V^{\perp}, \quad V'_{\xi}, \quad \mathscr{V},$$

where V'_{ξ} represents the (infinite) family of one-dimensional subspaces of $\mathscr{V}$ consisting of lines that are perpendicular to V. None of these lines can coincide with $\mathbf{u}$ because then $\mathbf{u}$ and $\mathbf{v}$ will be perpendicular. And none of these lines can be invariant under $R^{\alpha}_{\mathbf{u}}$ because α is not a multiple of π. This leaves only $\{0\}$ and $\mathscr{V}$ as spaces that are invariant under both rotations. Therefore, Π_{nat} is irreducible. □

In view of the identifications in Proposition 4 and the result in Proposition 3, we have the following:

Proposition 6 *Assume case C_2 of* (27) *holds. Then the deviatoric representation Π_{dev} of the group G generated by $R^{\alpha}_{\mathbf{u}}$ and $R^{\beta}_{\mathbf{v}}$ is irreducible if none of the following holds:*

1. $\alpha = \pi/2$ *and* $|\mathbf{u}\cdot\mathbf{v}| = 1/\sqrt{2}$
2. $\alpha = 2\pi/3$ *and* $|\mathbf{u}\cdot\mathbf{v}| = 1/\sqrt{3}$
3. $\alpha = 2\pi/3$ *and* $|\mathbf{u}\cdot\mathbf{v}| = \sqrt{2/3}$

10.3.3 Cases C_3 and C_4

Here we show that the groups corresponding to the cases C_3 and C_4 always have reducible natural representations.

Proposition 7 *Assume that case C_3 or C_4 of* (27) *holds. Then the natural representation Π_{nat} of the group G generated by $R^{\alpha}_{\mathbf{u}}$ and $R^{\beta}_{\mathbf{v}}$ is reducible.*

Proof If $\alpha = \beta = \pi$, then the one-dimensional subspace $\text{span}\{\mathbf{u}\times\mathbf{v}\} \subset \mathscr{V}$ is invariant under $R^{\alpha}_{\mathbf{u}}$ and $R^{\beta}_{\mathbf{v}}$. Therefore, Π_{nat} is reducible.

If $\alpha \neq \pi$, $\beta = \pi$, and $\mathbf{u}\cdot\mathbf{v} = 0$. Then, since $\mathbf{u}$ and $\mathbf{v}$ are perpendicular, then the one-dimensional subspace $\text{span}\{\mathbf{u}\} \subset \mathscr{V}$ is invariant under $R^{\alpha}_{\mathbf{u}}$ and $R^{\pi}_{\mathbf{v}}$. Therefore, Π_{nat} is reducible. □

It follows immediately from the above result that symmetry groups corresponding to the cases C_3 and C_4 do not imply isotropy, as Theorem 1 does not apply. Moreover, by Theorem 2 we know if the natural representation of a symmetry group G is reducible, then there exists an anisotropic elasticity tensor C having G as a symmetry group.

10.4 The main result in 3D

We are now in a position to apply Theorem 3 to characterize the symmetry groups that imply isotropy of an elasticity tensor in three-dimensional elasticity. To that purpose, we introduce:

Definition 1 Let $\mathbf{u}$ and $\mathbf{v}$ be non-collinear unit vectors and let $R^{\alpha}_{\mathbf{u}}$ and $R^{\beta}_{\mathbf{v}}$ be rotations by angles α and β about $\mathbf{u}$ and $\mathbf{v}$, respectively. We say $\{R^{\alpha}_{\mathbf{u}}, R^{\beta}_{\mathbf{v}}\}$ is a *special rotation pair* if at least one of the following conditions holds:

α	β	$\lvert\mathbf{u}\cdot\mathbf{v}\rvert$
$\pi/2$	$\pi/2$	0
$\pi/2$	$2\pi/3$	$1\sqrt{3}$
$\pi/2$	π	$1/\sqrt{2}$
$2\pi/3$	$2\pi/3$	$1/3$
$2\pi/3$	π	$1/\sqrt{3}$ or $\sqrt{2/3}$
φ	π	0
π	π	–

Table 1 The symmetry group generated by a pair of rotations $R_{\mathbf{u}}^{\alpha}$ and $R_{\mathbf{v}}^{\beta}$ implies isotropy if α, β and $\mathbf{u}\cdot\mathbf{v}$ are other than the special cases tabulated here. Here φ stands for an arbitrary angle.

1. $\alpha=\beta=\pi/2$ and $\mathbf{u}\cdot\mathbf{v}=0$
2. $\alpha=\pi/2, \beta=2\pi/3$ and $|\mathbf{u}\cdot\mathbf{v}|=1/\sqrt{3}$
3. $\alpha=\beta=2\pi/3$ and $|\mathbf{u}\cdot\mathbf{v}|=1/3$
4. $\alpha=\pi/2$, $\beta=\pi$, and $|\mathbf{u}\cdot\mathbf{v}|=1/\sqrt{2}$
5. $\alpha=2\pi/3$, $\beta=\pi$, and $|\mathbf{u}\cdot\mathbf{v}|=1/\sqrt{3}$ or $|\mathbf{u}\cdot\mathbf{v}|=\sqrt{2/3}$
6. $\alpha=2\pi/3$, $\beta=\pi$, and $|\mathbf{u}\cdot\mathbf{v}|=\sqrt{2/3}$
7. $\alpha\neq\pi$, $\beta=\pi$, and $\mathbf{u}\cdot\mathbf{v}=0$
8. $\alpha=\beta=\pi$

If $\{R_{\mathbf{u}}^{\alpha}, R_{\mathbf{v}}^{\beta}\}$ is *not* a special rotation pair, then we say it is a *generic rotation pair*, or *generic* for short. For instance, if **u** and **v** are orthogonal and $\alpha=\beta=\pi/3$, then $\{R_{\mathbf{u}}^{\alpha}, R_{\mathbf{v}}^{\beta}\}$ is generic.

Theorem 10 *Let* $\mathsf{C}: Sym \to Sym$ *be linear and not necessarily self-adjoint. Suppose that the identity*

$$Q\mathsf{C}[E]Q^T=\mathsf{C}[QEQ^T] \quad \text{for all } E\in Sym \tag{28}$$

holds for $Q=R_{\mathbf{u}}^{\alpha}$ *and* $Q=R_{\mathbf{v}}^{\beta}$*, where* $\{R_{\mathbf{u}}^{\alpha}, R_{\mathbf{v}}^{\beta}\}$ *is a generic rotation pair. Then* C *is isotropic.*

Proof Let G be the group generated by $R_{\mathbf{u}}^{\alpha}$ and $R_{\mathbf{v}}^{\beta}$. Clearly, the relation (28) holds for every $Q\in G$ also.

If α and β are both different from π (case C_1), then we know by Proposition 2 that the representation Π_{nat} of G is irreducible. Moreover, since none of the conditions 1, 2, and 3 of Definition 1 hold, we know by Proposition 3 that Π_{dev} is also irreducible. Hence, the result follows from Theorem 3.

If $\alpha\neq\pi$, $\beta=\pi$, and **u** and **v** are not perpendicular (case C_2), we know by Proposition 5 that the representation Π_{nat} of G is irreducible. Moreover, since none of the conditions 4, 5, and 6 of Definition 1 hold, we know by Proposition 6 that Π_{dev} is also irreducible. Thus, the result follows from Theorem 3.

Finally, not having conditions 7 and 8 of Definition 1, rules out cases C_3 and C_4 in which the representation Π_{nat} of G would be *reducible*. □

Table 1 summarizes the content of the theorem in a tabular form.

11 Polar elasticity revisited

In this section we show that under the same hypothesis as that of Theorem 10, a linear mapping $\mathsf{C}: Lin \to Lin$ is isotropic.

We have already dealt with irreducibility of natural and deviatoric representations of groups generated by two rotations. Here we complete the analysis by characterizing the irreducibility of the skew representation.

Proposition 8 *Assume that case C_1 or C_2 of* (27) *holds. Then the skew representation Π_{skw} of the group G generated by $R_{\mathbf{u}}^{\alpha}$ and $R_{\mathbf{v}}^{\beta}$ is irreducible.*

Proof Recall that any group generated by two rotations satisfying case C_2 corresponds to a group in case C_1. Therefore, it is sufficient to prove that a group G generated by two rotations $R_{\mathbf{u}}^{\alpha}$ and $R_{\mathbf{v}}^{\beta}$ satisfying case C_1 has irreducible Π_{skw}. Hence, G will be a group generated by $R_{\mathbf{u}}^{\alpha}$ and $R_{\mathbf{v}}^{\beta}$ with $\mathbf{u}$ and $\mathbf{v}$ non-collinear, and $\alpha \neq \pi$ and $\beta \neq \pi$.

Let $Q = R_{\mathbf{u}}^{\alpha}$, and let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis of $\mathscr{V}$ with $\mathbf{e}_1 = \mathbf{u}$. Without loss of generality we may assume,

$$Q\mathbf{e}_1 = \mathbf{e}_1, \quad Q\mathbf{e}_2 = (\cos\alpha)\mathbf{e}_2 + (\sin\alpha)\mathbf{e}_3, \quad Q\mathbf{e}_3 = -(\sin\alpha)\mathbf{e}_2 + (\cos\alpha)\mathbf{e}_3. \tag{29}$$

Also, let $\{E_1, E_2, E_3\}$ be given by

$$E_1 = \mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1, \quad E_2 = \mathbf{e}_1 \otimes \mathbf{e}_3 - \mathbf{e}_3 \otimes \mathbf{e}_1, \quad E_3 = \mathbf{e}_2 \otimes \mathbf{e}_3 - \mathbf{e}_3 \otimes \mathbf{e}_2, \tag{30}$$

be an orthogonal basis of *Skew*. Then we have:

$$\begin{aligned}
\Pi_{\text{skw}}(Q)[E_1] &= (\cos\alpha)E_1 + (\sin\alpha)E_2, \\
\Pi_{\text{skw}}(Q)[E_2] &= -(\sin\alpha)E_1 + (\cos\alpha)E_2 \\
\Pi_{\text{skw}}(Q)[E_3] &= E_3.
\end{aligned}$$

This shows that on $Skew^{\dagger}$, $\Pi_{\text{skw}}^{\dagger}(Q)$ has eigenvalues $\{e^{\pm i\alpha}, 1\}$ corresponding to the eigenvectors $\{E_1 \mp iE_2, E_3\}$. Moreover, since $\alpha \neq \pi$, the eigenvalues are distinct.

Let $V_1 = \text{span}\{E_3\}$ and $V_2 = \text{span}\{E_1, E_2\}$ and note that the subspaces of *Skew* invariant under $\Pi_{\text{skw}}(Q)$ are $\big\{\{0\}, V_1, V_2, Skew\big\}$. For $\tilde{Q} = R_{\mathbf{v}}^{\beta}$, we may analogously get an orthonormal basis $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ of $\mathscr{V}$, an orthogonal basis $\{F_1, F_2, F_3\}$ of *Skew*, and the invariant subspaces $\big\{\{0\}, V_1', V_2', Skew\big\}$.

We shall show that $\Pi_{\text{skw}}(Q)$ and $\Pi_{\text{skw}}(\tilde{Q})$ share no invariant subspaces other than $\{0\}$ and *Skew*. For this, it suffices to show that V_1 and V_1' do not coincide; if they do, then there exists a nonzero $\xi \in \mathbf{R}$ such that $F_3 = \xi E_3$, that is

$$\mathbf{f}_2 \otimes \mathbf{f}_3 - \mathbf{f}_3 \otimes \mathbf{f}_2 = \xi\big(\mathbf{e}_2 \otimes \mathbf{e}_3 - \mathbf{e}_3 \otimes \mathbf{e}_2\big). \tag{31}$$

Applying both sides of (31) to $\mathbf{f}_1$ gives, $\xi\langle\mathbf{f}_1, \mathbf{e}_3\rangle\mathbf{e}_2 - \xi\langle\mathbf{f}_1, \mathbf{e}_2\rangle\mathbf{e}_3 = 0$, which in turn implies that $\langle\mathbf{f}_1, \mathbf{e}_2\rangle = \langle\mathbf{f}_1, \mathbf{e}_3\rangle = 0$. Therefore, it must be the case that $\mathbf{f}_1$ and $\mathbf{e}_1$ are collinear. This however contradicts the standing assumption that $\mathbf{e}_1 = \mathbf{u}$ and $\mathbf{f}_1 = \mathbf{v}$ are non-collinear. We conclude that the only subspaces of *Skew* invariant under both $\Pi_{\text{skw}}(R_{\mathbf{u}}^{\alpha})$ and $\Pi_{\text{skw}}(R_{\mathbf{v}}^{\beta})$ are the zero subspace and *Skew* itself. Hence, the representation Π_{skw} of the group G generated by $R_{\mathbf{u}}^{\alpha}$ and $R_{\mathbf{v}}^{\beta}$ is irreducible. □

Combining Proposition 8 with Theorem 5, we arrive at the following interesting result:

Proposition 9 *Let $\mathsf{W} : Skew \to Skew$ be linear and not necessarily self-adjoint. Let $R_{\mathbf{u}}^{\alpha}$ and $R_{\mathbf{v}}^{\beta}$ be rotations of angles α and β about non-collinear vectors $\mathbf{u}$ and $\mathbf{v}$ respectively, where $\mathbf{u}$, $\mathbf{v}$, α, and β satisfy either C_1 or C_2. Suppose that the identity*

$$Q\mathsf{W}[A]Q^T = \mathsf{W}[QAQ^T] \quad \textit{for all } A \in Lin, \tag{32}$$

holds for $Q = R_{\mathbf{u}}^{\alpha}$ and $Q = R_{\mathbf{v}}^{\beta}$. Then, $\mathsf{W} \in Sph(Skew)$.

Remark 7 The above representation result for linear mappings on *Skew* was obtained in [7] and [23] when the identity (32) holds for all $Q \in Orth$.

Theorem 11 *Let $\mathsf{C} : Lin \to Lin$ be linear and not necessarily self-adjoint. Suppose that the identity*

$$Q\mathsf{C}[A]Q^T = \mathsf{C}[QAQ^T] \quad \text{for all } A \in Lin \tag{33}$$

holds for $Q = R_{\mathbf{u}}^{\alpha}$ and $Q = R_{\mathbf{v}}^{\beta}$, where $\{R_{\mathbf{u}}^{\alpha}, R_{\mathbf{v}}^{\beta}\}$ is a generic rotation pair. Then,

$$\mathsf{C}[A] = a(\mathrm{tr}A)I + bA + cA^T, \quad \text{for all } A \in Lin \tag{34}$$

where a, b, and c are real constants.

Proof Let G be the group generated by $R_{\mathbf{u}}^{\alpha}$ and $R_{\mathbf{v}}^{\beta}$. Then, the identity (33) holds for all $Q \in G$. We know by Propositions 2 (if in case C_1) or Proposition 5 (if in case of C_2) that the representation Π_{nat} of G is irreducible. Moreover, Proposition 3 (if in case C_1) or Proposition 6 (if in case C_2) imply that the representation Π_{dev} of G is irreducible. Finally, Proposition 8 implies that the representation Π_{skw} of G is irreducible. Therefore, the result follows from Theorem 4. □

Remark 8 An elementary derivation of the representation (34), assuming the symmetry group is the full *Orth*, is given in [14, Chapter VII]. Alternative derivations appear in [6] and [23], (also see references therein). This is generalized in [2] to tensors in *n*-dimensions, where it is observed that only a certain finite subset of elements of *Orth* suffice. The novelty of Theorem 11 is in that only an arbitrary pair of generic rotations suffices to reach the same conclusion, albeit in three dimensions.

References

1. M. M. Carroll. Must elastic materials be hyperelastic? *Mathematics and Mechanics of Solids*, 4:369–376, 2009.
2. Carl de Boor. A naive proof of the representation theorem for isotropic, linear asymmetric stress-strain relations. *J. Elasticity*, 15(2):225–227, 1985.
3. Gianpietro Del Piero. Some properties of the set of fourth-order tensors, with application to elasticity. *J. Elasticity*, 9(3):245–261, 1979.
4. Gianpietro Del Piero. Representation theorems for hemitropic and transversely isotropic tensor functions. *J. Elasticity*, 51(1):43–71, 1998.
5. William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.
6. Zhong Heng Guo. The representation theorem for isotropic, linear asymmetric stress-strain relations. *J. Elasticity*, 13(2):121–124, 1983.
7. Zhong Heng Guo and P. Podio-Guidugli. A concise proof of the representation theorem for linear isotropic tensor-valued mappings of a skew argument. *J. Elasticity*, 21(3):317–320, 1989.
8. Morton E. Gurtin. A short proof of the representation theorem for isotropic, linear stress-strain relations. *J. Elasticity*, 4(3):243–245, 1974.
9. Morton E. Gurtin. *An introduction to continuum mechanics*, volume 158 of *Mathematics in Science and Engineering*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1981.
10. Paul R. Halmos. *Finite-dimensional vector spaces*. The University Series in Undergraduate Mathematics. D. Van Nostrand Co., Inc., Princeton-Toronto-New York-London, 1958. 2nd ed.
11. Yong Zhong Huo and Gianpietro Del Piero. On the completeness of the crystallographic symmetries in the description of the symmetries of the elastic tensor. *J. Elasticity*, 25(3):203–246, 1991.
12. Gordon James and Martin Liebeck. *Representations and characters of groups*. Cambridge University Press, New York, second edition, 2001.
13. Jovo P. Jaric. On the representation of symmetric isotropic 4-tensors. *J. Elasticity*, 51(1):73–79, 1998.
14. Harold Jeffreys. *Cartesian Tensors*. Cambridge University Press, 1931.

15. V. V. Jikov, S. M. Kozlov, and O. A. Oleĭnik. *Homogenization of differential operators and integral functionals*. Springer-Verlag, Berlin, 1994. Translated from the Russian by G. A. Yosifian [G. A. Iosif'yan].
16. C. S. Jog. A concise proof of the representation theorem for fourth-order isotropic tensors. *J. Elasticity*, 85(2):119–124, 2006.
17. James K. Knowles. On the representation of the elasticity tensor for isotropic materials. *J. Elasticity*, 39(2):175–180, 1995.
18. L. C. Martins. The representation theorem for linear, isotropic tensor functions revisited. *J. Elasticity*, 54(1):89–92, 1999.
19. L. C. Martins and P. Podio Guidugli. A new proof of the representation theorem for isotropic, linear constitutive relations. *J. Elasticity*, 8(3):319–322, 1978.
20. G. A. Maugin. On the structure of the theory of polar elasticity. *R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci.*, 356(1741):1367–1395, 1998.
21. Martin Ostoja-Starzewski. *Microstructural randomness and scaling in mechanics of materials*. CRC Series: Modern Mechanics and Mathematics. Chapman & Hall/CRC, Boca Raton, FL, 2008.
22. G. C. Papanicolaou and S. R. S. Varadhan. Boundary value problems with rapidly oscillating random coefficients. In *Random fields, Vol. I, II (Esztergom, 1979)*, volume 27 of *Colloq. Math. Soc. János Bolyai*, pages 835–873. North-Holland, Amsterdam, 1981.
23. K. A. Pericak-Spector and Scott J. Spector. On the representation theorem for linear, isotropic tensor functions. *J. Elasticity*, 39(2):181–185, 1995.
24. Issai Schur. Neue begründung der theorie der gruppencharaktere. *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, Part 1*, pages 406–432, 1905.
25. Salvatore Torquato. *Random heterogeneous materials*, volume 16 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York, 2002. Microstructure and macroscopic properties.
26. C. Truesdell. The mechanical foundations of elasticity and fluid dynamics. *J. Rational Mech. Anal.*, 1:125–171, 173–300, 1952.
27. C. Truesdell. *Continuum mechanics. I. The mechanical foundations of elasticity and fluid dynamics*. International Science Review Series, Vol. 8. Gordon and Breach Science Publishers, New York, 1966.
28. C. Truesdell. *The Elements of Continuum Mechanics*. Springer-Verlag, 1966.
29. C. Truesdell and W. Noll. *The non-linear field theories of mechanics*. Springer-Verlag, Berlin, third edition, 2004. Edited and with a preface by Stuart S. Antman.
30. H. Xiao. General irreducible representations for constitutive equations of elastic crystals and transversely isotropic elastic solids. *J. Elasticity*, 39(1):47–73, 1995.