Sensitivity of estimating parameters in an SIR epidemic model to observed data at different times

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Abstract

We consider an inverse problem of estimating parameters in a simple SIR model of disease dynamics. We analyze the sensitivity of the solution of the inverse problem to data observed at different times. We do this using Hyper-differential sensitivity analysis (HDSA), which seeks to assess the sensitivity of the solution of an optimization problem to problem data [1]. The analysis shows that the estimation of different model parameters shows sensitivity to different measurement times. This also indicates key observation times when observations should be collected.

1 Introduction

Consider the problem of estimating the parameters β and γ in

$$\begin{aligned} S' &= -\frac{\beta}{N} SI, \\ I' &= \frac{\beta}{N} SI - \gamma I, \\ R' &= \gamma I, \end{aligned}$$

where $N = S + I + R$ be the total population (which here is a conserved quantity). Given measurements of $I(t)$ at t_i, $i = 1, \ldots, n$, we can estimate the parameter vector $\boldsymbol{m} = [\beta\, \gamma]^\top$ by solving an inverse problem. Let's denote the vector of measurements by $\boldsymbol{y}$ and by $f(\boldsymbol{m})$ the parameter-to-observable map

$$\boldsymbol{f}(\boldsymbol{m}) = \begin{bmatrix} I(t_1; \boldsymbol{m}) \\ I(t_2; \boldsymbol{m}) \\ \vdots \\ I(t_n; \boldsymbol{m}) \end{bmatrix}.$$

Note that evaluating $f(\boldsymbol{m})$ requires solving the SIR system with parameter vector $\boldsymbol{m}$ and extracting solution values at the measurement times. The inverse problem of estimating the parameter vector $\boldsymbol{m}$ is as follows:

$$\min_{\boldsymbol{m}} J(\boldsymbol{m}) := \|\boldsymbol{f}(\boldsymbol{m}) - \boldsymbol{y}\|^2. \tag{1.1}$$

We seek to understand the sensitivity of the solution of the inverse problem to measurements y_j, $j = 1, \ldots, n$. To do so, we consider the modified objective function

$$J(\boldsymbol{m}) = \|\boldsymbol{f}(\boldsymbol{m}) - \tilde{\boldsymbol{y}}(\boldsymbol{\theta})\|^2, \quad \text{where} \quad \tilde{y}_i(\boldsymbol{\theta}) = (1 + \theta_i) y_i, \quad i = 1, \ldots, n,$$

and $\boldsymbol{y}$ is a fixed vector of measurement data. The elements θ_i, $i = 1, \ldots, n$ define perturbations of a nominal data vector $\boldsymbol{y}$. Let $\boldsymbol{m}^* = \boldsymbol{m}^*(\boldsymbol{\theta})$ be the solution of the inverse problem

$$\min_{\boldsymbol{m}} J(\boldsymbol{m}, \boldsymbol{\theta}) := \|\boldsymbol{f}(\boldsymbol{m}) - \tilde{\boldsymbol{y}}(\boldsymbol{\theta})\|^2.$$

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We will perform HDSA by computing

$$\mathbf{D} = \frac{\partial \boldsymbol{m}^*}{\partial \boldsymbol{\theta}}\Big|_{\boldsymbol{\theta}=\bar{\boldsymbol{\theta}}},$$

where $\bar{\boldsymbol{\theta}} = \mathbf{0}$. Computing $\mathbf{D}$ is facilitated by differentiating through the optimality condition

$$\boldsymbol{g}(\boldsymbol{m}^*(\boldsymbol{\theta}), \boldsymbol{\theta}) = \mathbf{0}, \tag{1.2}$$

where $\boldsymbol{g}$ denotes gradient of J with respect to $\boldsymbol{m}$. We have used the Implicit Function Theorem (assuming sufficient regularity) to write (1.2), which will hold in a neighborhood of $(\bar{\boldsymbol{m}}^*, \bar{\boldsymbol{\theta}})$, and express the solution of the inverse problem as a function of $\boldsymbol{\theta}$ in a neighborhood of $\bar{\boldsymbol{\theta}}$. Here $\bar{\boldsymbol{m}}^*$ is the solution of the inverse problem, corresponding to $\bar{\boldsymbol{\theta}}$; i.e, $\bar{\boldsymbol{m}}^* = \boldsymbol{m}^*(\bar{\boldsymbol{\theta}})$.

Taking partial derivative with respect to θ_i through (1.2) gives

$$\underbrace{\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{m}}}_{2\times 2} \underbrace{\frac{\partial \boldsymbol{m}^*}{\partial \theta_i}}_{2\times 1} + \underbrace{\frac{\partial \boldsymbol{g}}{\partial \theta_i}}_{2\times 1} = \underbrace{\mathbf{0}}_{2\times 1}, \quad i = 1, \ldots, n, \tag{1.3}$$

Note that $\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{m}}$ is the Hessian $\mathbf{H}$ of J (with respect to $\boldsymbol{m}$). We can write (1.3) in the compact form

$$\mathbf{H}\,\mathbf{D} = -\mathbf{B},$$

where $\mathbf{B}$ is defined by

$$B_{ji} = \frac{\partial^2 J}{\partial \theta_i \partial m_j}, \quad i = 1, \ldots, n, \; j = 1, 2.$$

In the present analysis, $\mathbf{B}$ and $\mathbf{H}$ are computed at $(\boldsymbol{m}, \boldsymbol{\theta}) = (\bar{\boldsymbol{m}}^*, \bar{\boldsymbol{\theta}})$.

2 Simulation setup

We consider the SIR model above in the interval $[0, 20]$. We assume "true" parameter vector $\boldsymbol{m} = [\beta\,\gamma]^\top = [2\,1]^\top$. The initial conditions are assumed to be $S(0) = 9900$, $I(0) = 100$, and $R(0) = 0$. In the numerical computations, we considered the fraction of infected people, $I(t)/N$ when defining measurements and the parameter-to-observable map, instead of $I(t)$; see Figure 1.

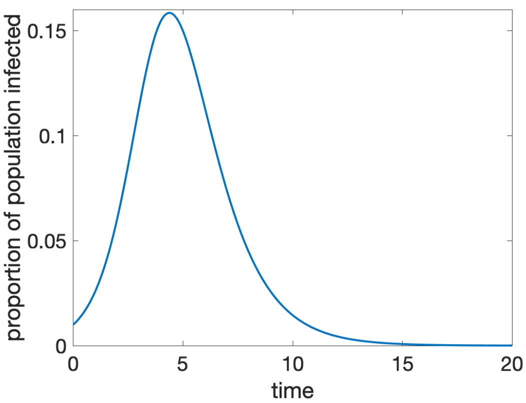

Figure 1: The proportion of the population that is infected over time.

We synthesized noisy measurements, with 2.5% noise, at 56 equally spaced observation times in the interval $[0, 12]$. This inverse problem can be solved easily to get good estimates of the parameter vector $\boldsymbol{m}$. Numerically, we computed parameter estimates $\beta^* = 2.0116$ and $\gamma^* = 1.0106$. To provide further insight into this simple inverse problem, we present a plot of the objective function J in Figure 2.

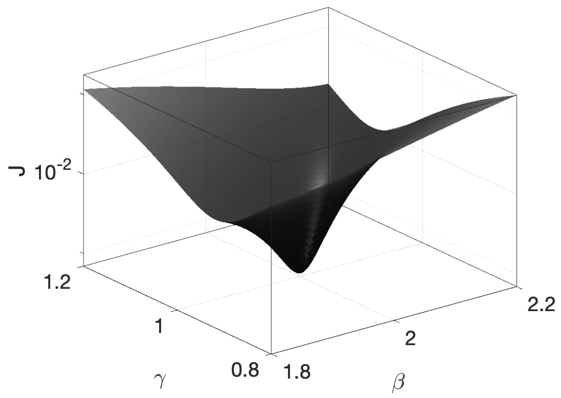

Figure 2: The objective function $J(\boldsymbol{m}) = J(\beta, \gamma)$ in a neighborhood of the solution to the inverse problem.

3 Results of sensitivity analysis and discussion

We computed the sensitivity of the solution of the inverse problem, $\boldsymbol{m}^* = [\beta^* \, \gamma^*]^\top$, given by

$$\boldsymbol{m}^* = \underset{\boldsymbol{m}}{\operatorname{argmin}}\, J(\boldsymbol{m}),$$

with respect to data perturbation vector $\boldsymbol{\theta}$. The plots in Figure 3 presents the results of the sensitivity analysis.

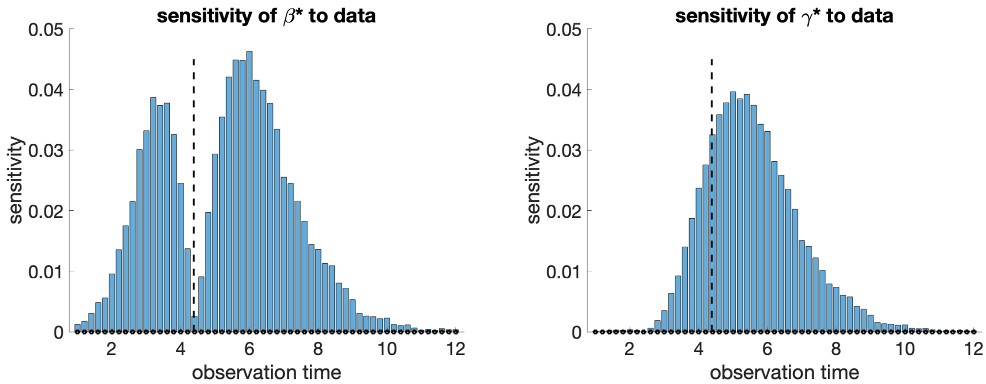

Figure 3: The sensitivity of the estimated β^* and γ^* with respect to θ_i, $i = 1, \ldots, n$. Here we have taken $n = 56$ equally spaced measurements. The dashed vertical line indicates the peak of the epidemic when the number of infecteds is at its maximum.

We draw the following conclusions:

1. The estimated β^* is sensitive to measurements leading up to and following the peak of the infected population curve, but not at or near the peak. We see slightly higher sensitivity to the measurements following the peak. We note that β^* is not very sensitive to measurements very early or later in time as we approach a steady state.
2. The sensitivities for the estimated γ^* are quite different. The observations become important as we are near the peak of the epidemic curve and we see high sensitivities to data following the peak. As before, early measurements and those later in time, as we approach the steady state, are not as important. Lack of sensitivity to early observations is more pronounced in this case.

References

[1] Hart, Joseph, Bart van Bloemen Waanders, and Roland Herzog. "Hyper-Differential Sensitivity Analysis of Uncertain Parameters in PDE-Constrained Optimization.". Preprint. 2019. `https://arxiv.org/abs/1909.07336`